Algebra 2 Notes

Shaun Carter

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For Sarah, who proves every day that math equals love.

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Contents

Chapter 1

Functions

1.1 Sets

A ______ is a collection of mathematical objects. In this class, it will almost always be a collection of _____________. Sets are usually represented by _________________ variables.

Sets can be defined as a list of values, or by using a rule, notated by __________________.

Example 1 If set A contains only the values 1, 2, 3, 6, 8 and 9, then

If set B contains all values greater than or equal to 6, then

Note that either : or | can be used in set notation. If reading aloud, say "______________".

$x \in S$ says that the value x _______________________ the set S, or x is _____ S.
$x \notin S$ says the opposite: the value x is _________ the set S.

Example 2 Using the definitions of A and B above, write $\in$ or $\notin$.

$1 \quad A$	$4 \quad A$	$6 \quad A$	$7 \quad A$	$5.9 \quad A$	$8.1 \quad A$
$1 \quad B$	$4 \quad B$	$6 \quad B$	$7 \quad B$	$5.9 \quad B$	$8.1 \quad B$

Symbols for Special Sets

Typed	Written	Name	Description
$\varnothing$			The set that contains no elements at all.
$\mathbb{N}$			The set of numbers[1] used for counting. $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{Z}$			The set containing all the natural numbers, their negative counterparts, and 0. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$			The set of numbers which can be written as fractions using integers. Real numbers not in this set (including π) are called _____________.
$\mathbb{R}$			The set of _____ numbers which can be placed on the number line.

[1] Many mathematicians would say the natural numbers also include 0. If you want unambiguous terms, you can use *positive integers* to exclude 0, and *nonnegative integers* include 0.

Combining Sets

$A \cap B$ is the ________________ of A and B. It is a set that contains all the elements that are in **both** A **and** B.

$A \cup B$ is the _________ of A and B. It is a set that contains all the elements that are in **either** A **or** B.

$A \backslash B$ is the ____________________ of A and B. It is a set that contains all the elements that are **in** A **but not in** B.

Example 3 $C = \{1, 5, 7, 10\}$ and $D = \{4, 5, 6, 7, 8\}$

$C \cap D =$ $C \cup D =$

$C \backslash D =$ $D \backslash C =$

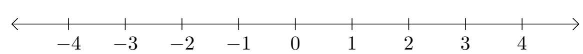
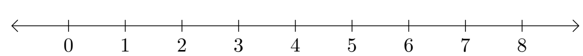
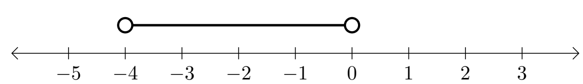
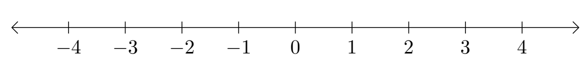
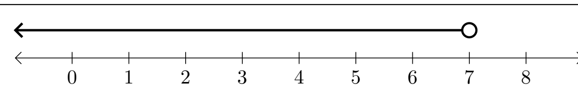
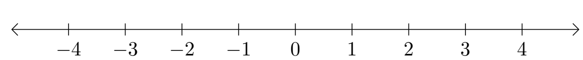
Interval Notation

An ___________ is a special type of set which contains all real numbers between a _________ _________, a, and an __________ _________, b.

$[a, b]$ represents an interval with bounds which are ____________. (a, b) represents an interval with bounds which are _____________. $(a, b]$ and $[a, b)$ can be used when the bound types are mixed.

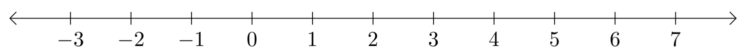
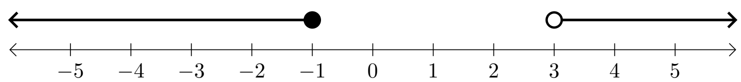
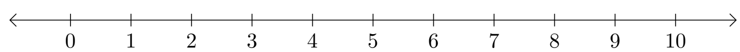
On number lines and graphs, an included bound is represented by a ___________________, and an excluded bound is represented by an ___________________.

Example 4

Interval	Set Notation	Real Number Line
$[-2, 3)$		
	$\{x \mid 1 < x \leq 6\}$	
		
	$\{x : x \geq -2\}$	
		
$(-\infty, \infty)$		

If a set consists of ________________ intervals, the ________ symbol can be used to include them in the same set.

Examples:

Interval Notation	Real Number Line
$(-3,1)\cup[4,7]$	
	
$[1,2)\cup(3,4]\cup[6,\infty)$	

If a set contains all real numbers ________________ values, there are multiple options for notating the set.

Example 5 The set containing all real numbers except 2 and 5 is

Interval Notation	Set Notation	Set Difference

Comparing Sets

If every element in a set U is also in another set V, then we can write $U \subset V$. We say that U is a __________ of V, and that V is a ____________ of U. We can also say that V ___________ U.

Example 6 Let $A=\{-1,2,3,4\}$ and $B=\{-1,2,3,4,5.5,7\}$.

Set Relation	T/F	Reason
$A\subset B$		
$B\subset A$		
$A\subset \mathbb{N}$		
$A\subset \mathbb{Z}$		
$B\subset \mathbb{Z}$		
$A\subset[-1,4)$		
$B\subset[-1,7]$		
$[-1,4)\subset[-1,7]$		
$\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$		

1.2 Introduction to Functions

A ___________ is a collection of ordered pairs which represents a relationship between two sets of real numbers. Each ordered pair is typically labeled as (x, y).

The first set, which contains all x-values, is called the __________. The second set, which contains the y-values, is called the _____________.

A ____________ is a particular type of relation. In a function, each value in the domain is ____________ related to a value in the codomain. In other words, for each x, there is _________________ y related to it.

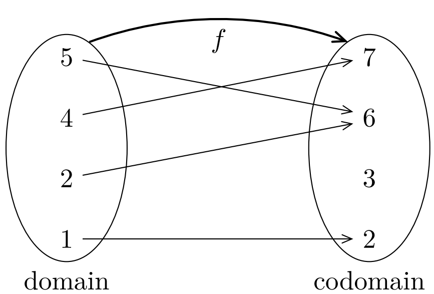
To say that a function f relates a domain A and a codomain B, we write

which can be read aloud as _____________________________.

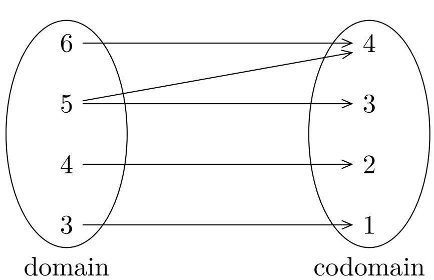
The relation between x and y is written as

The _________ (or image) of a function is the __________ of the _____________ that contains the values that are actually produced by the function. We can think of the domain as the _________ of the function, and the range as the ___________ of the function.

Example 1 Find the domain, codomain and range of the function, and find the value of $f(x)$ for each value x in the domain.

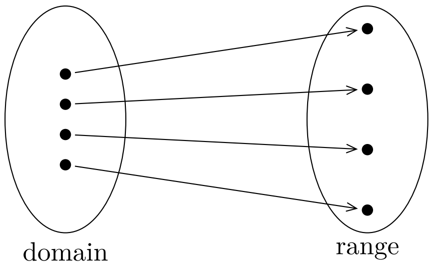


Example 2 Explain why the following relation is **not** a function.



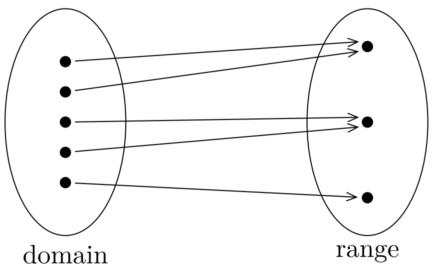
One-to-One and Many-to-One Functions

For every function, each x-value in the domain maps to a unique y-value in the range. It is not necessarily true that each y-value is mapped to by a unique x-value.



In a ___________________________, each y-value in the range is only mapped to by one x-value in the domain.

Equivalently, $f(a) = f(b)$ if and only if $a = b$.



In a ______________________________, at least one y-value in the range mapped to by more than one x-value in the domain.

Equivalently, there is an a and b in the domain such that $f(a) = f(b)$, but $a \neq b$.

Function Evaluation

To ____________ a function means to determine the value of $f(a)$ for a given value a in the domain. If a is not in the domain, then $f(a)$ is said to be ______________.

Example 3 The function f is defined by the table shown.

x	$f(x)$
-3	4
-2	3
-1	0
0	1
1	-1
2	5
3	2

The domain of f is

The range of f is

The relation type of f is

$f(2)$

$f(4)$

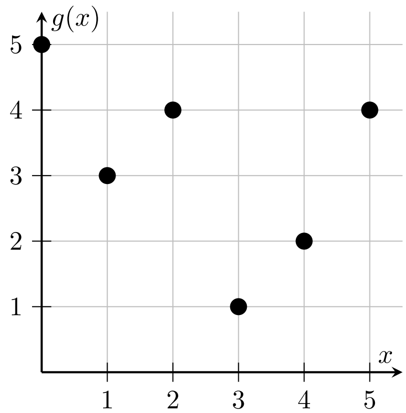
$f(-2) + f(2)$

$2f(-3) - 5f(0)$

$f\left(f\left(1\right)\right)$

$f\left(f\left(f\left(-2\right)\right)\right)$

Example 4 The function g is defined by the graph shown.



The domain of g is

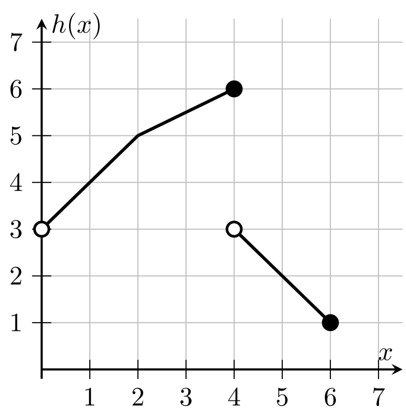
The range of g is

The relation type of g is

$g(3)$ $g(1.5)$

$g\left(g\left(g\left(0\right)\right)\right)$

Example 5 The function h is defined by the graph shown.



The domain of h is

The range of h is

The relation type of h is

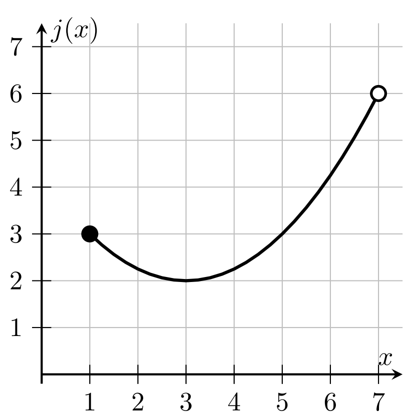
$h(4)$ $h(1.5)$

$h(0)$ $h(2.5)$

$h\left(g\left(1\right)\right)$

$g\left(h\left(1\right)\right)$

Example 6 The function j is defined by the graph shown.



The domain of j is

The range of j is

The relation type of j is

$j(3)$ $j(7)$

$j(2)$ $j(6)$

1.3 Inverse Functions and Solving Equations

Suppose we have a ____________, which consists of a collection of ordered pairs in the form (x, y). Its ________________________ is the relation whose ordered pairs are switched to be (y, x).

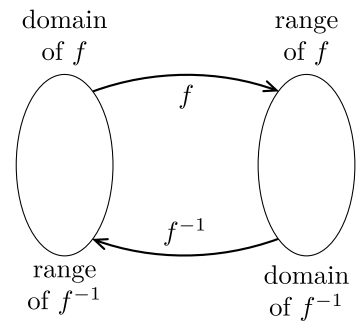
Recall that a ______________ is a special type of relation. If the ________________________ of a ______________ is also a ______________, it is called the _________________________.

If a function is denoted _____, its inverse function, if it exists, is denoted ________.

Properties of Inverse Functions

If function f has the inverse function f^{-1}, then

- The inverse function of ________ is _____.
- The ____________ of f^{-1} is identical to the ___________ of f.
- The ___________ of f^{-1} is identical to the ____________ of f.
- As the inverse function results from switching the x and y values, the ____________ of $y = f(x)$ and $y = f^{-1}(x)$ are _________________, or ______________________ of each other across the line __________.



Condition for Inverse Functions

Suppose function f is defined by the following table, and suppose f^{-1} is its inverse function.

x	1	2	3
$f(x)$	7	8	7

What is $f^{-1}(8)$?

What is $f^{-1}(7)$?

Because $f^{-1}(7)$ has ____________ values, f^{-1} is ______________________. This has happened because f is a ____________________ function. Therefore,

Theorem

A function f has an ____________________ f^{-1}
if and only if f is a ______________ function.

Example 1 The function f is defined by the table shown.

x	$f(x)$
-3	4
-2	3
-1	0
0	1
1	-1
2	2

The domain of f is

The range of f is

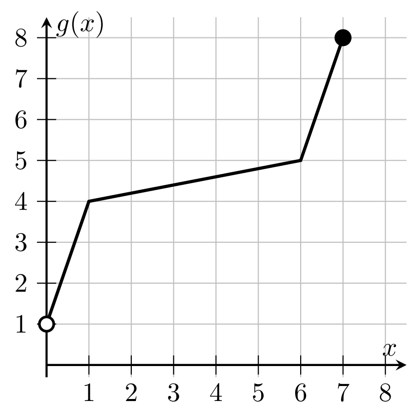
The inverse function f^{-1} ______ exist because the function is ____________.

The domain of f^{-1} is

The range of f^{-1} is

x	$f^{-1}(x)$

Example 2 The function g is defined by the graph shown.



The domain of g is

The range of g is

The inverse function g^{-1} ______ exist because the function is ____________.

The domain of g^{-1} is

The range of g^{-1} is

$g(1)$ $g(6)$ $g(7)$

$g^{-1}(4)$ $g^{-1}(5)$ $g^{-1}(8)$

Solving Equations using Inverse Functions

Recall that we can use ______________________ to solve equations. If an equation contains a ________________________, we can use its ____________________ in the same way to solve the equation.

If a solution ________, this method will ensure that it is _________. If the equation requires applying the ____________________ to a value for which it is ____________, then the equation has ______________.

Example 3 Solve the following equations using the table defining f.

x	-3	-2	-1	0	1	2	3
$f(x)$	4	3	0	1	-1	5	2

$2f(x+3)-4=6$

$$\frac{f(5x)-1}{3}=2$$

Solving Equations with no Inverse Function

If an equation contains a ______________________________, it may still be possible to solve the equation. However, the solution may not be __________.

Example 4 Solve the following equations using the table defining g.

x	-3	-2	-1	0	1	2	3
$g(x)$	3	2	1	3	2	1	3

$3g(x-5)+2=8$

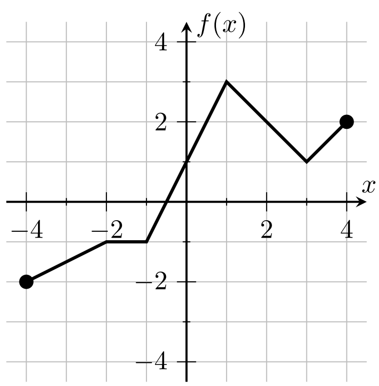
$$\frac{g(x)+7}{2}=5$$

1.4 Transformations

A ____________________ is a ________ which, when applied to a _______________________, produces an __________ of the figure with each point changed in a prescribed way.

In this class we'll consider transformations of ___________ of functions and how they change the function _________________.

For the following examples, we'll use the function f, as defined by this graph and table:



x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1.5	-1	-1	1	3	2	1	2

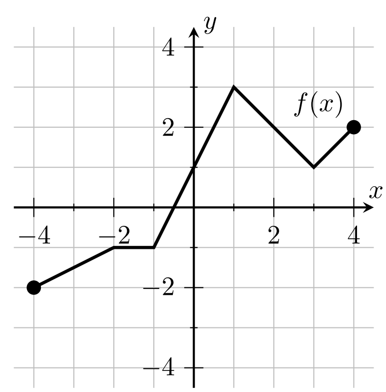
Reflections

A ______________ is a transformation which creates a ___________________ across a ________________________. Each point in the image remains the ____________________ from this line, but on the __________________.

Example 1

$$g(x) = f(-x)$$

x									
$-x$	-4	-3	-2	-1	0	1	2	3	4
$f(-x)$ $g(x)$									



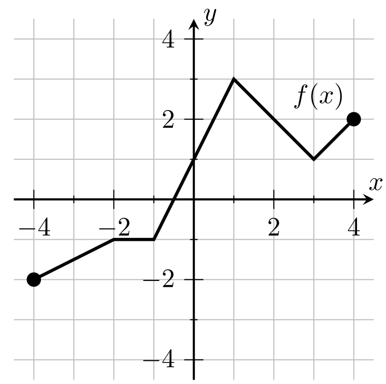
Each x-value ______________________________. Each y-value ___________________.

The graph has been _____________________________________.

Example 2

$$g(x) = -f(x)$$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									
$-f(x)$ $g(x)$									



Each x-value ____________________. Each y-value ________________________________.

The graph has been _______________________________________.

Stretches and Compressions

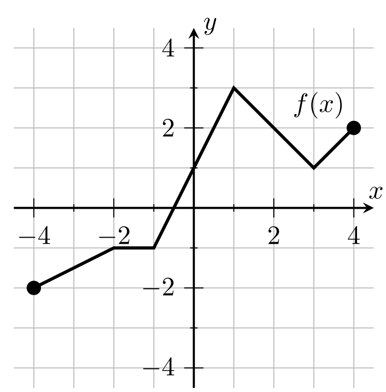
A ____________ or __________________ is a transformation where each point's distance from a ______________ is multiplied by a __________________.

If each point gets ____________________ the fixed line, the transformation is a ____________. If each point gets ______________ the fixed line, the transformation is a __________________.

Example 3

$$g(x) = f(2x)$$

x									
$2x$	-4	-3	-2	-1	0	1	2	3	4
$f(2x)$ $g(x)$									



Each x-value ___________________________. Each y-value ____________________.

The graph has been ____ by a factor of ____.

Example 4

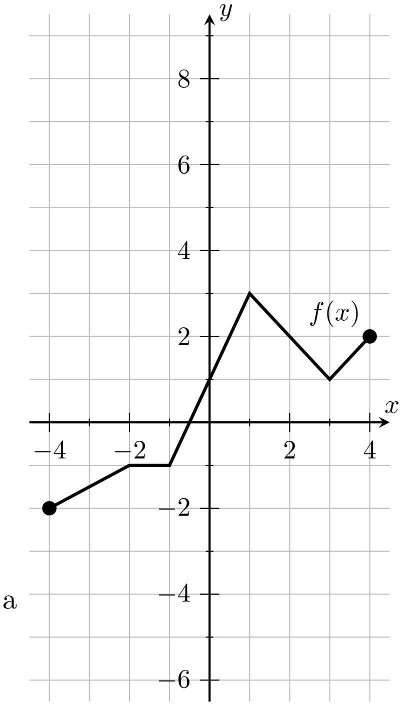
$$g(x) = 3f(x)$$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									
$3f(x)$ $g(x)$									

Each x-value ____________________.

Each y-value _________________________.

The graph has been _____________________________________ by a factor of _____.



Translations

A _______________, or _________, is a transformation where every point in the image is moved ___________________________ in ___________________________.

A translation can be ___________________, or __________________, or a combination of directions.

Example 5

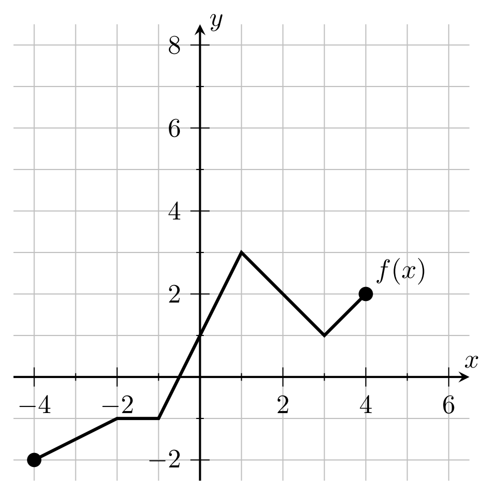
$$g(x) = f(x - 2) + 5$$

x									
$x - 2$	-4	-3	-2	-1	0	1	2	3	4
$f(x-2)$									
$f(x-2)+5$ $g(x)$									

Each x-value __________________________.

Each y-value __________________________.

The graph has been ______________________________ and _________________.



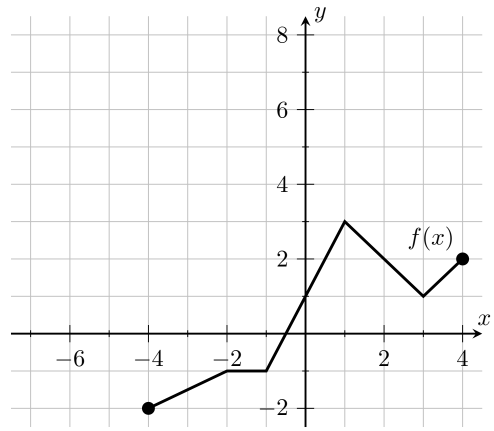
Combining Transformations

Example 6 $$g(x) = 2f\left[-(x+3)\right] + 2$$

x									
$x+3$									
$-(x+3)$	-4	-3	-2	-1	0	1	2	3	4
$f\left[-(x+3)\right]$									
$2f\left[-(x+3)\right]$									
$2f\left[-(x+3)\right]+2$ $g(x)$									

The graph has been:

- ____________ across the __________,
- _____________ from the __________ by a factor of ____,
- _______________ by ____ units, and
- ______________ by ____ units.



When listing transformations for the usual form $g(x) = A \cdot f\left[n(x-h)\right] + k$, translations should always be listed _________ reflections and dilations.

Summary of Transformations

$y = A \cdot f(x)$	reflect across the x-axis if ___________________ stretch from the x-axis by a factor of $\lvert A\rvert$ if ____________ compress toward the x-axis by a factor of $\frac{1}{\lvert A\rvert}$ if _________________
$y = f(n \cdot x)$	reflect across the y-axis if ___________________ stretch from the y-axis by a factor of $\frac{1}{\lvert n\rvert}$ if _________________ compress toward the y-axis by a factor of $\lvert n\rvert$ if ___________
$y = f(x-h) + k$	translate $\lvert h\rvert$ units right if _________________, left if ____________ translate $\lvert k\rvert$ units up if _________________, down if ____________

Chapter 2

Linear Functions and Equations

2.1 Linear Functions

A ____________________ is a function with the algebraic form

where m and b are constants.

This corresponds to the ____________________________ of a linear relation, named because the graph of the function is a _________________, where m is the _________ of the line and b is its ________________.

If a function is defined by an __________________, the function is evaluated by ________________ the appropriate value from the ___________ into the rule, and calculating the result.

Example 1 $f : [-3, 6) \to \mathbb{R}$, where $f(x) = -2x + 8$.

$f(2)$ $f(5)$ $f(-3)$

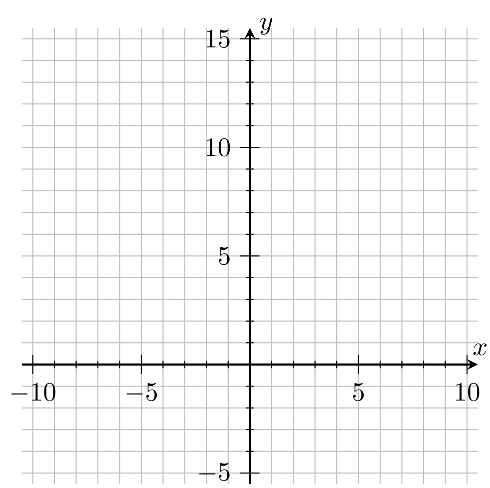
$f(7)$ $f(-1.25)$

Graphing Functions

A useful tool to ____________ a function is its __________. The graph consists of a _________[1] drawn on a ______________________, or ______________________[2].

If x is in the ___________ of the function f, then the ________ $(x, f(x))$ will be part of the curve.

Example 2 Plot the function f from Example 1 on the coordinate plane to the right.



[1]Even if it's a straight line, it's still called a "curve".
[2]Named after the 17th Century French philosopher, René Descartes.

Implied Domains

It is common practice to state only the rule of a function, without stating the domain. In these cases, it is reasonable to assume the ____________________, which is the _____________________ domain for which the function can be ______________.

For a ____________________, the implied domain is ___________________________, because

.

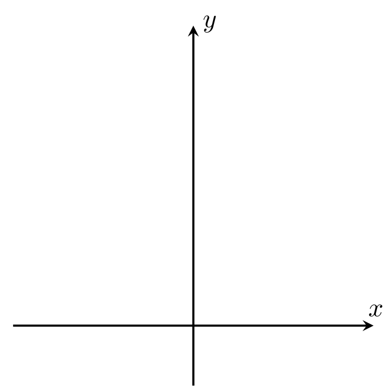
Sketching Linear Functions

A ___________ is a version of a graph that shows only the _____________________. In the case of a linear function, the information that should be included is:

shape of curve	
x-intercept	
y-intercept	
endpoints	

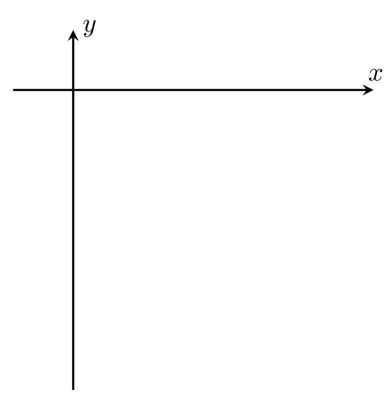
Example 3 Sketch $f(x) = 4x + 6$.
Shape:
x-intercept:

y-intercept:
endpoints:



Example 4 Sketch $g(x) = -\frac{1}{2}x + 1$ on the domain $[2, \infty)$.
Shape:
x-intercept:

y-intercept:
endpoints:

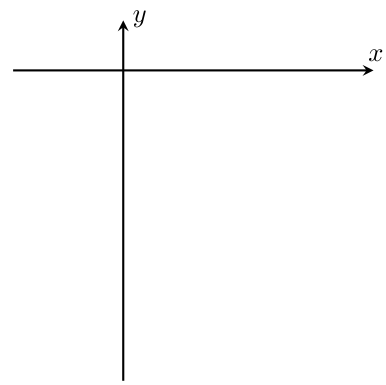


Note that it is a good idea to include at least two points so the slope of the line is clear.

Example 5 Find the range of $h : (-1, 5] \to \mathbb{R}$ where $h(x) = -2x - 3$, and sketch the graph of $h(x)$.

Shape:

x-intercept:



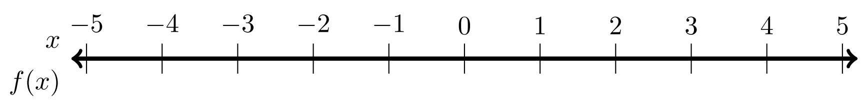
y-intercept:

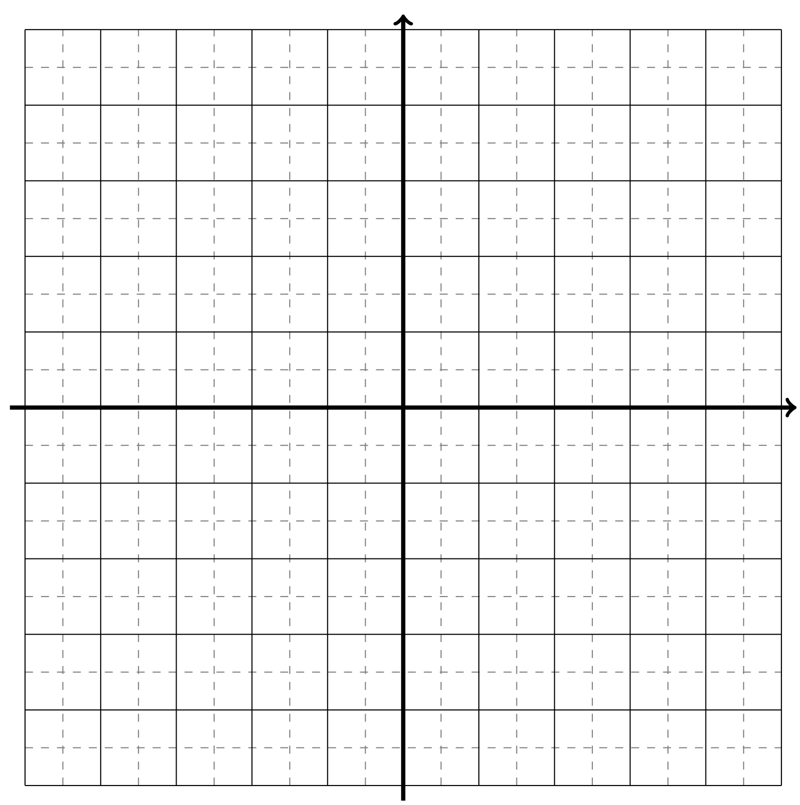
endpoints:

The Linear Parent Function

For any given function, its ____________________ is the simplest function of the same type.

parent function
domain
range
relation type
x-intercept
y-intercept
slope





Transformations of Linear Functions

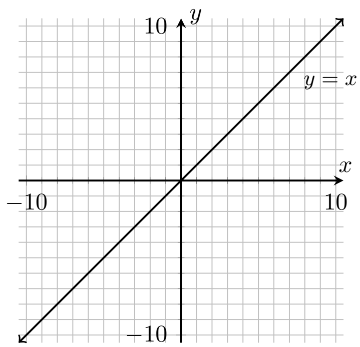
Recall that $g(x) = Af(x) + k$ represents a ____________ or __________________ from the x-axis if $|A| \neq 1$, a ______________ across the x-axis if A is negative, and a _______________ up or down.

If we let $A = m$, $k = b$, and $f(x) = x$, then $g(x) = mx + b$, the general form of linear functions. This gives us the following result:

Theorem

Every ____________________, $g(x) = mx + b$, is the result of a ____________________ applied to its ____________________, $f(x) = x$.

Example 6 Write the transformations needed to obtain $g(x) = -2x + 5$ from its parent function.



Example 7 The graph of $y = x$ is compressed by a factor of 4 toward the x-axis, shifted 8 units left and shifted 7 units down. What is resulting function in slope-intercept form?

Transformations do not need to be applied only to the parent function, but can be used with any function.

Example 8 The function $f : [-2, 5) \to \mathbb{R}$, where $f(x) = 2x + 4$, is reflected across the x-axis and shifted 3 units right. Find the resulting function g in the form $g(x) = mx + b$.

Find the new domain:

Find the new rule:

Example 9 Find the transformations required to transform $f(x) = 3x + 2$ to $g(x) = -6x + 5$.

2.2 Inverses of Linear Functions

Recall that a function has an ____________________ if and only if it is a ________________________.

Since non-constant ________ functions are ______________ (think about why this is true) we can conclude the following:

Theorem

Each ________________, $f(x) = mx + b$, where ________,

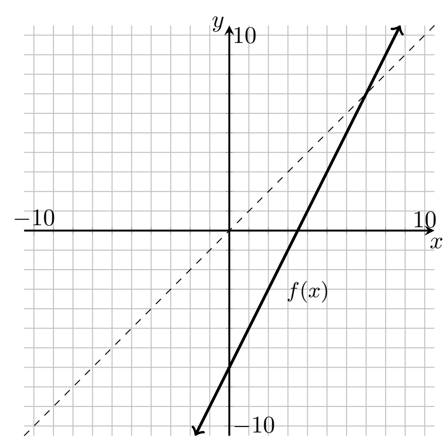
has an __________________.

Finding the Inverse Function

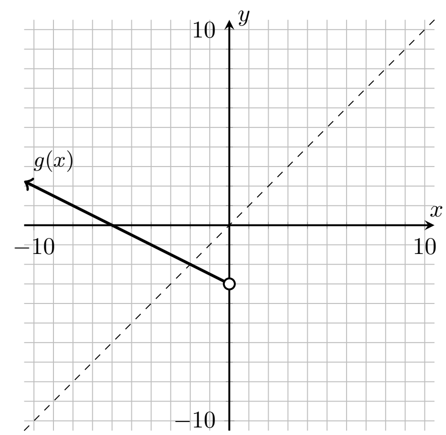
Recall that the __________ of a relation results from ____________________________. For an algebraically defined function, we can find the inverse by following these steps:

1. Replace $f(x)$ with ____.
2. Rewrite the equation by ______________________.
3. Rearrange the equation so that ______________.
4. Check that y is a __________; if so, replace y with __________.

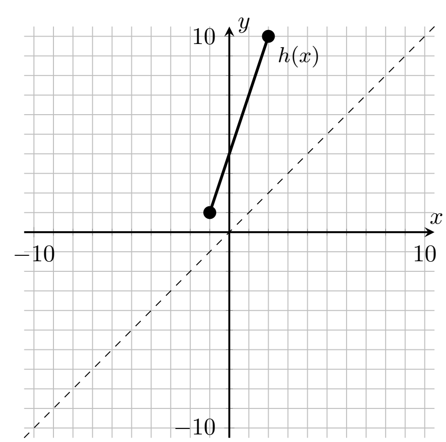
Example 1 Find the inverse function of $f(x) = 2x - 7$.



Example 2 Find the inverse of $g : (-\infty, 0) \to \mathbb{R}$, where $g(x) = -\frac{1}{2}x - 3$.



Example 3 Find the inverse of $h : [-1, 2] \to \mathbb{R}$, where $h(x) = 3x + 4$.



2.3 Systems of Linear Equations

A ____________________________ is a collection of multiple ______________ containing multiple ______________, or variables. A ___________ to the system consists of values for the unknowns that satisfy all of the equations ___________________.

Example 1 Verify that $x = 2$, $y = 5$, $z = -3$ is a solution to

$$\begin{cases} x + y + z = 4 \\ 2x - y - z = 2 \\ x + 3y + 2z = 11 \end{cases}$$

Solving Systems of Two Equations Using Substitution

1. Choose one equation, and _______________ it to __________ one unknown.
2. _______________ this equation into the other and _________ for the remaining unknown.
3. _______________ this solution into the first rearranged equation to find the first unknown.
4. State the final solution for ________ unknowns, by stating each value separately or together as an ordered pair.

Example 2

$$\begin{cases} x + 2y = 10 & (1) \\ 2x - 3y = 6 & (2) \end{cases}$$

Example 3

$$\begin{cases} 2x - 3y = -11 & (1) \\ 3x - y = 8 & (2) \end{cases}$$

Solving Systems of Two Equations Using Elimination

1. Choose one unknown you want to have ___________________________. Make this true by ______________ the equations by appropriate values.
2. _____________ this unknown by __________ the equations.
3. _________ for the remaining unknown.
4. _______________ this solution into one of the original equations to find the first unknown.
5. State the final solution for ________ unknowns.

Example 4

$$\begin{cases} 4x + 5y = -5 & (1) \\ -2x - y = 7 & (2) \end{cases}$$

Example 5

$$\begin{cases} 3x + 4y = 2 & (1) \\ 2x - 5y = 9 & (2) \end{cases}$$

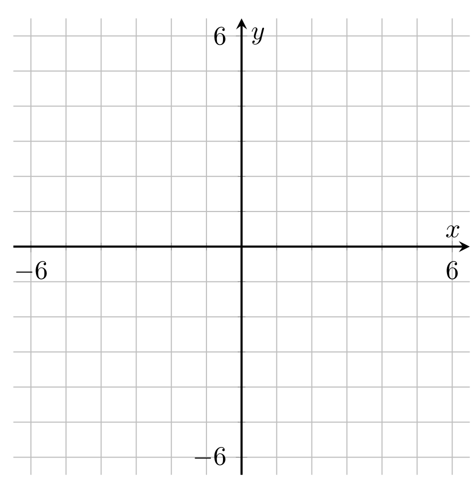
Solving Systems of Two Equations Using Graphs

Recall that when an equation is graphed, each ________ on the curve represents an __________________ that ____________ the equation.

Suppose both equations of a system are graphed on the ________________. Any points of ________________ will represent ordered pairs which satisfy ________ equations. This is exactly what we're looking for as a ___________ to the system.

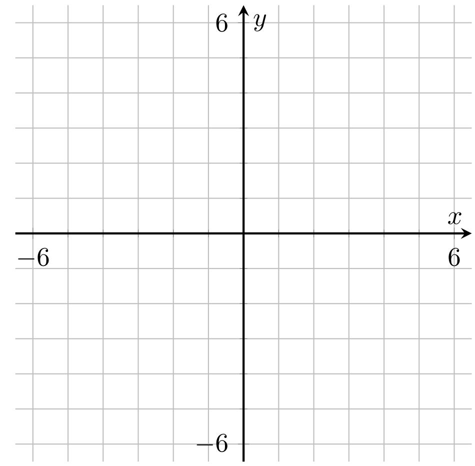
Example 6

$$\begin{cases} y = x - 4 & (1) \\ x + y = 2 & (2) \end{cases}$$



Example 7

$$\begin{cases} x - 2y = 6 & (1) \\ \quad\quad y = 4x + 4 & (2) \end{cases}$$

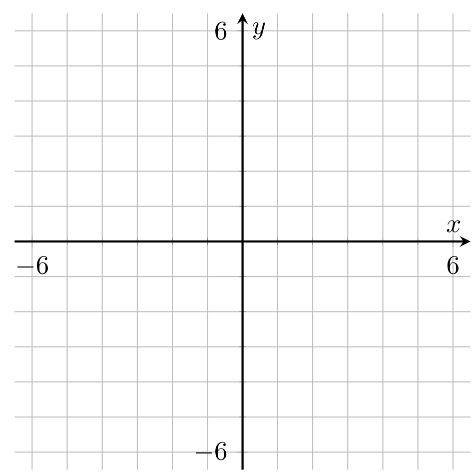


Types of Solutions to Systems of Linear Equations

Each of the earlier example systems have ____________________________. This is not always the case. Linear systems may instead have ________________________________, or have ________________.

Example 8 Algebraically find the nature of the solution to this system. Represent it with a graph.

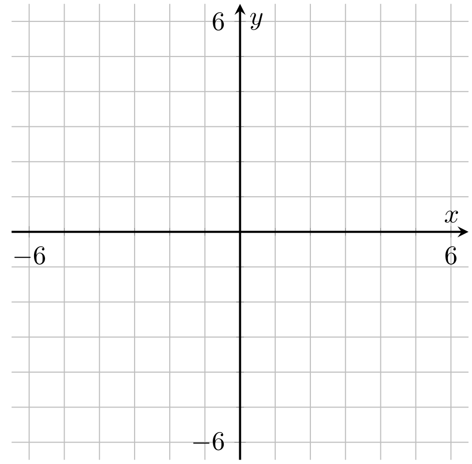
$$\begin{cases} 2x - y = 4 & (1) \\ 6x - 3y = 12 & (2) \end{cases}$$



These equations are _______________, because they _______________________ at the same time. The graphical representation has _____________________________________ because the lines are _______________.

Example 9 Algebraically find the nature of the solution to this system. Represent it with a graph.

$$\begin{cases} x + 2y = -2 & (1) \\ 2x + 4y = 8 & (2) \end{cases}$$



These equations are ________________, because they _____________________ at the same time. The graphical representation has ____________________ because the lines are ___________.

Systems of Three Linear Equations

For a system of _____________________ with _____________________, we can use the same techniques to find a solution.

1. Use ________________ or ______________ to remove one unknown from the system.
2. Solve for the remaining two unknowns.
3. Use the partial solution to solve for the removed unknown. State the complete solution.

Example 10 Using substitution:

$$\begin{cases} x + y + z = 6 & (1) \\ 2x - y + 3z = 11 & (2) \\ -x + 3y + 4z = 8 & (3) \end{cases}$$

Example 11 Using elimination:

$$\begin{cases} x + y + z = 6 & (1) \\ 2x - y + 3z = 11 & (2) \\ -x + 3y + 4z = 8 & (3) \end{cases}$$

2.4 Linear Regression

Functions are often used for ____________ real-world situations. Typically, the value of an ___________________________ is used as an input for the function, whose output is used to predict the value of a _________________________.

Scatter Plots

A ________________ is a plot used to visualize the relationship between two-variables, where each data point is treated as an _________________ and plotted as a ________ on a plane.

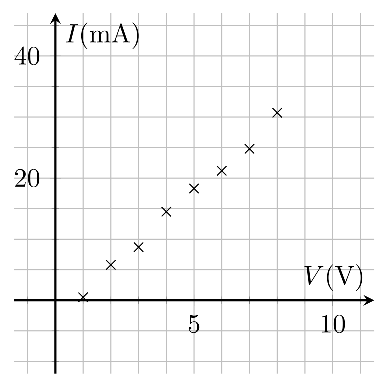
Visually inspecting a scatter plot can help decide whether a ___________________ is an appropriate model for a given set of data.

The independent variable is placed on the ___________________, and the dependent variable is placed on the _________________.

Example 1 A voltage source is placed in an electronic circuit. For various voltages, the current in the circuit is measured. The following results are recorded:

V (V)	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
I (mA)	0.5	5.8	8.7	14.5	18.3	21.2	24.8	30.7

Note that voltage, V, is measured in volts, V, and current, I, is measured in milliampere, mA.



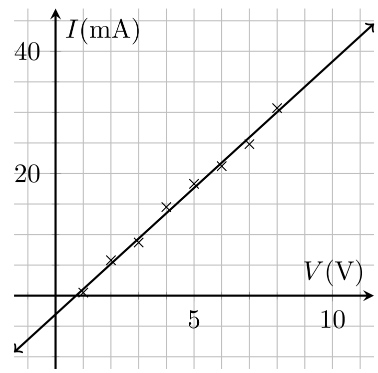
Regression

The process of _________ a function to a set of _______ in order to ________________ the association between variables is called _______________. When the modeling function is linear, it is called _______________________.

Since a linear function has the form ______________________, linear regression means choosing values for _____ and ____ in order to fit the data as well as possible.[3] We will be using _______________ to find these values for us.

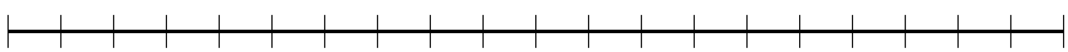
[3]You may think "as well as possible" is very vague. If so, you're right! The details of what this means are not important for Algebra 2, but they will be *very* important if you take a Statistics class in the future.

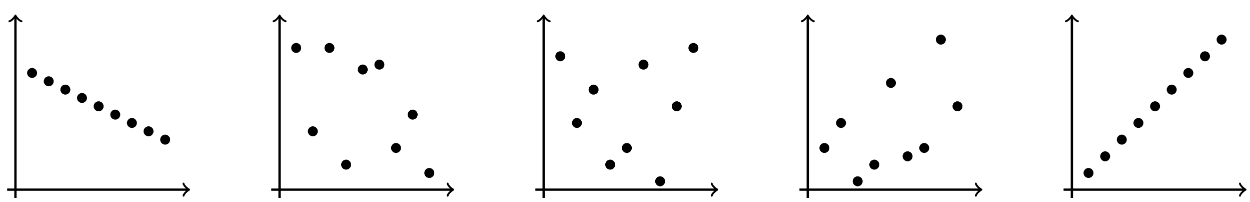
Example 2 For the electronic circuit example,



The Correlation Coefficient

The ____________________________, denoted by ____, is a quantity that measures the _____________ and _____________ of the linear association between two variables. r is in the interval ___________.





Example 3 For the electronic circuit example,

The Coefficient of Determination

The ___________________________________, denoted by ______ is a measure of how well a regression line, or curve, fits the provided data.[4] For _______________________ (but not other types of regression) it is the ___________ of the correlation coefficient, so _____________. Its value is in the interval _________.

Example 4 For the electronic circuit example,

[4] A statistics class would teach you that R^2 is the proportion of the variation in the dependent variable which is explained by the model. Don't worry if that doesn't make any sense yet!

Making Predictions

There are two types of predictions that we can make using a regression model.

________________ means predicting values _____________ the values in the data. If the model is a good fit for the data, then this can produce very reliable predictions.

Example 5 Estimate the current in the circuit when $V = 2.6\,\text{V}$.

Example 6 Estimate the voltage that corresponds to a current of $I = 27.3\,\text{mA}$.

_________________ means predicting values ___________ the values in the data. You need to be careful when _________________, because it is very difficult to know how far the trend in the data continues outside of its range.

Example 7 Estimate the current in the circuit when $V = 0.3\,\text{V}$.

2.5 Piecewise Linear Functions

A ______________________ is a function which is defined by ________________, each applying to different parts of the __________.

Example 1 Evaluate each of the following using the function f.

$$f(x) = \begin{cases} 2x & -2 \leq x \leq 3 \\ 4 & 3 < x < 6 \\ -x + 9 & x \geq 6 \end{cases}$$

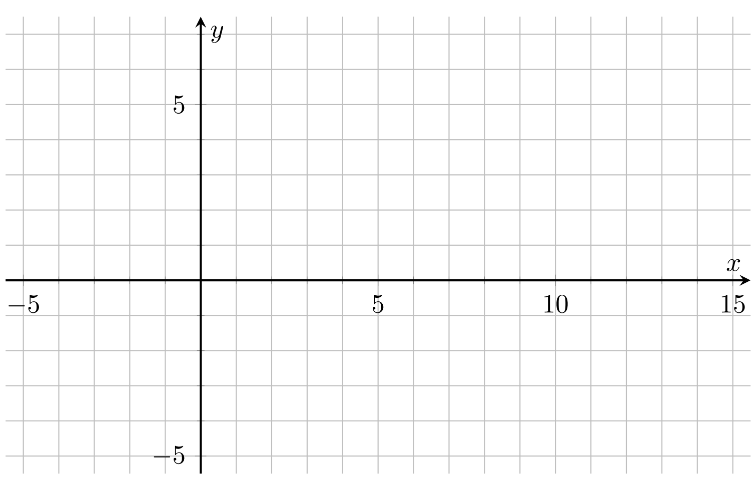
$f(1)$ $f(5)$ $f(8)$

$f(6)$ $f(3)$ $f(-3)$

A piecewise function can be ___________ by considering each rule separately, and plotting each on its own __________.

The __________ of the entire piecewise function is the ________ of the domains of the separate rules. Similarly, the _________ is the ________ of the __________ produced by each rule.

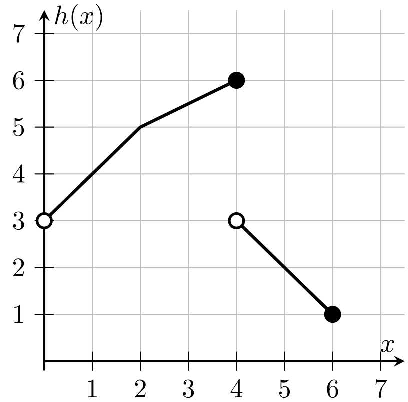
Example 2 For function f above, plot its graph and find its domain and range.



Domain:

Range:

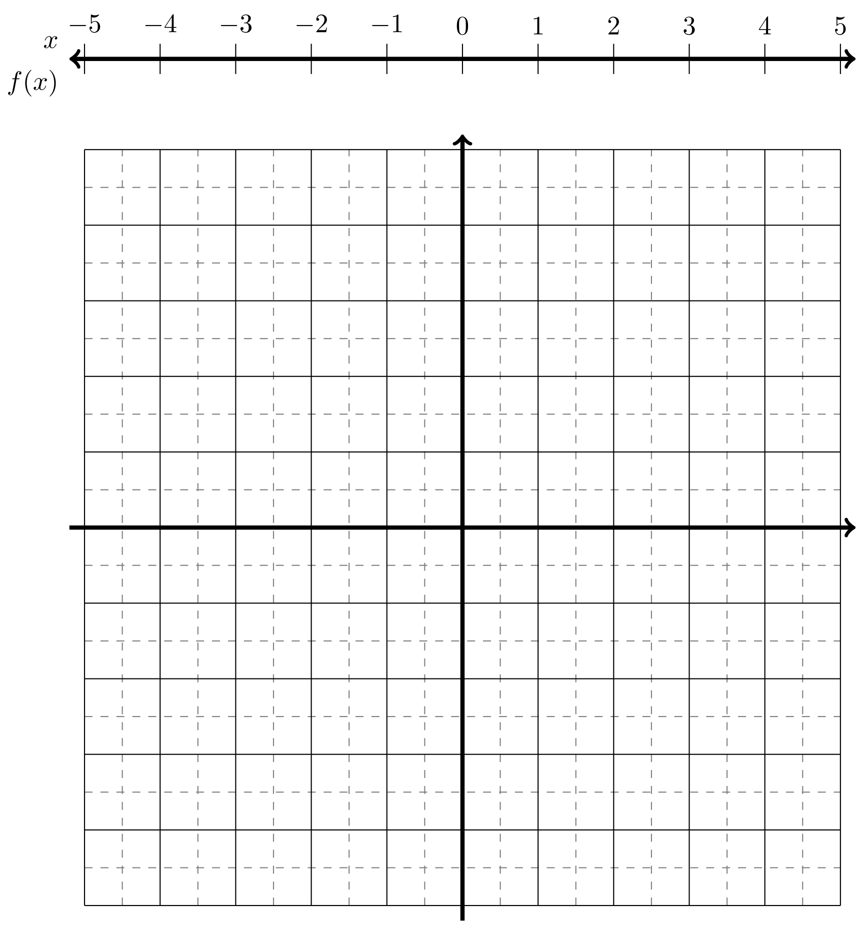
Example 3 Define h as a piecewise function.



The Absolute Value Parent Function

An important piecewise function is the ______________________________.

parent function
domain
range
relation type
x-intercept
y-intercept
vertex
slopes



Absolute Value Functions

By applying ____________________ to the parent function, we get the __________________ of the absolute value function:

- Graph is ___________ or opens _____ if A is ___________.
  Graph is ____________ or opens ________ if A is _____________.
- Graph has two _________ intervals, whose slopes are ________.
- Graph has a __________ at __________.

A sketch of an absolute value function should include:

shape of curve	
vertex	
x-intercepts	
y-intercept	
endpoints	

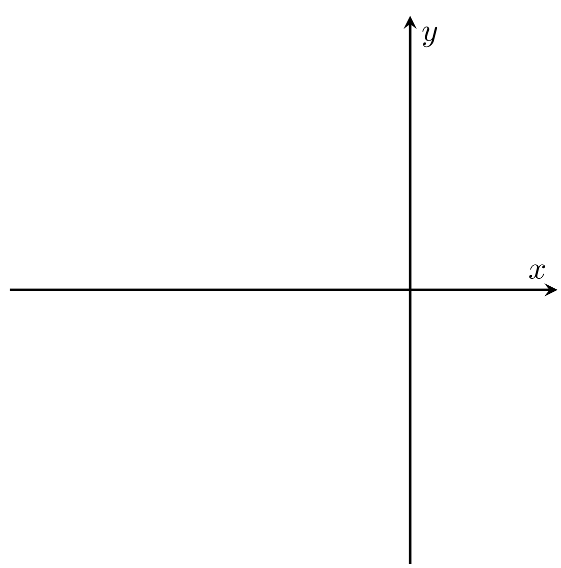
Example 4 Sketch $g(x) = -2\,|x+3| + 4$.

Orientation:

Slopes:

Vertex:

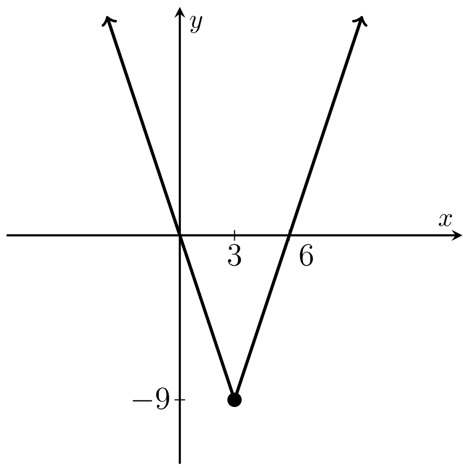
x-intercepts:



y-intercept:

endpoints:

Example 5 Find the function f represented by the following graph.



Orientation:

Slopes:

Vertex:

Example 6 Find the range of $f : [2, 9) \to \mathbb{R}$, where $f(x) = \frac{1}{2}|x-4| + 3$.

Example 7 Find the transformations required to transform $f(x) = 2|x-2| + 1$ to $g(x) = -3|x+1| + 6$.

Example 8 Express $f(x) = 5\,|x-4| + 7$ as a piecewise function.

Chapter 3

Quadratic Functions and Equations

3.1 Quadratics in Vertex Form

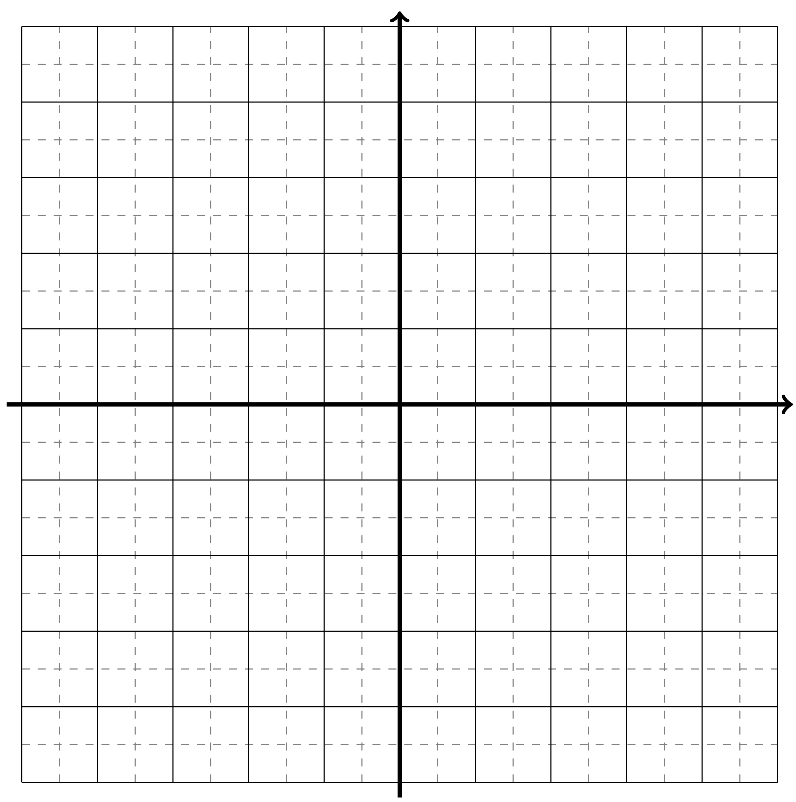
A ______________________________ is an expression which can be written in the form (with $a \neq 0$):

A ___________________________ is a function consisting of a quadratic expression. The three forms of these functions we usually consider are

The Quadratic Parent Function

parent function
domain
range
relation type
x-intercept
y-intercept
vertex

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$											



Solving Quadratic Equations Using Square Roots

A ________________________ is any equation which can be written with a _____________ _______________ on one side and ________ on the other. Note that this might not be the original form of the equation.

If an equation is written in __________________, it can be solved using __________________:

1. Rearrange the equation to __________ the quantity which is ____________.
2. Eliminate the square with a _________________. Consider both the ___________ and _____________ square roots.
3. Finish solving the equation by ____________ x.

Example 1 Solve $2(x-4)^2 - 5 = 13$

Example 2 Solve $-3(x+5)^2 + 7 = 7$

Example 3 Solve $(x+2)^2 - 7 = 0$

Example 4 Solve $2(x-6)^2 + 9 = 1$

Note that quadratic equations may have ________, _______, or _______ real[1] solutions.

[1]In an upcoming lesson, you will see that it is possible to get solutions that are not real numbers! For now, we're only considering the real numbers.

Graphing Quadratic Functions Using Vertex Form

By applying ____________________ to the quadratic parent function, we get the _________________ of a quadratic function:

$$f(x) = A(x - h)^2 + k$$

- Graph is ___________ or opens _____ if A is ___________.

  Graph is ____________ or opens ________ if A is ____________.

- _______ corresponds to a ___________ or ________________ from the x-axis.

- Graph has a __________ at __________.

A sketch of a quadratic function should include:

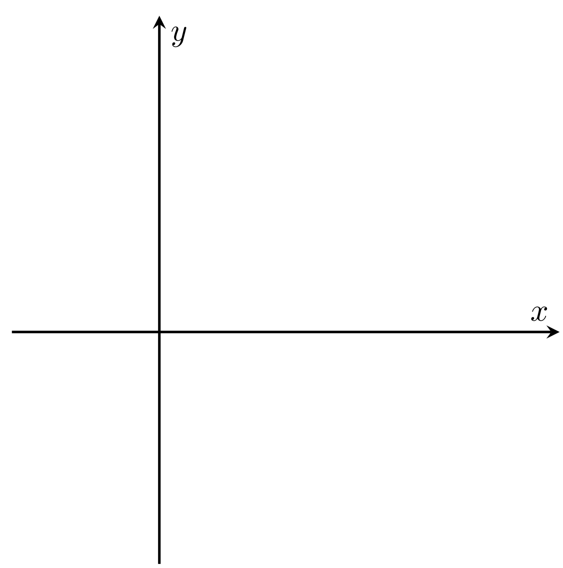
shape of curve	
vertex	
x-intercepts	
y-intercept	
endpoints	

Example 5 Sketch $f(x) = (x - 3)^2 - 4$.

Orientation:

Vertex:

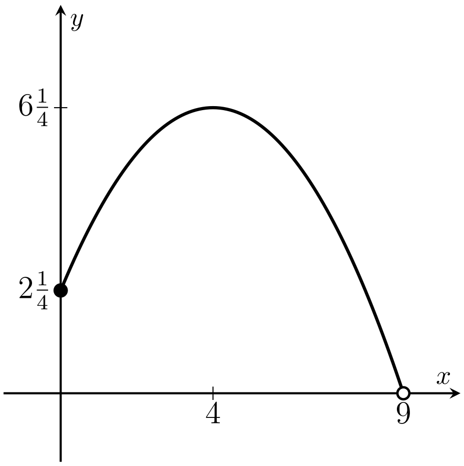
x-intercepts:



y-intercept:

endpoints:

Example 6 Find the function g represented by the following graph.



Vertex:

y-intercept:

Domain:

Example 7 Find the range of $h : [-3, 1] \to \mathbb{R}$, where $h(x) = -2(x+2)^2 + 7$.

Zeros, Roots, Solutions and x-Intercepts

These terms are related, but have subtly different meanings.

The ________ of an **expression** are the values which cause the expression to equal ________.

The ____________ of an **equation** are the values which cause the equation to be ________.

The _________ of a **function** are the input values which cause the output value to be ________.

The _________________ of a **graph** are the points where the curve __________________________.

Example 8

The ____________ of $(x-3)^2 - 4 = 0$ are

The _________ of $f(x) = (x-3)^2 - 4$ are

The ________ of $(x-3)^2 - 4$ are

The _________________ of the graph of $y = (x-3)^2 - 4$ are

3.2 Quadratics in Factored Form

The Zero Product Property

If _________, then ________ or ________ or ____________.

Equivalently, if the __________ of a set of __________ is _______, then at least one of the __________ is _______.

Quadratic Equations in Factored Form

Example 1 Solve $3x(x-5)=0$

Example 2 Solve $(x-4)(x+7)=0$

Example 3 Solve $(5x-2)(7x+4)=0$

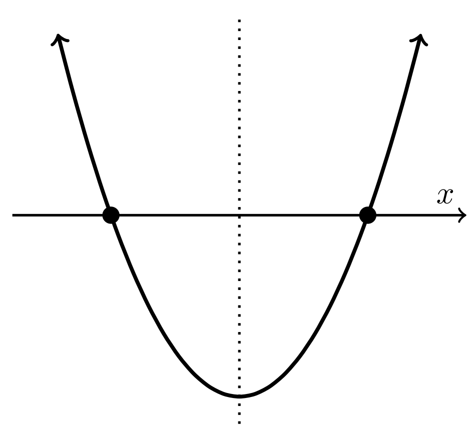
Example 4 Solve $(3x-8)^2=0$

Graphing Quadratic Functions in Factored Form

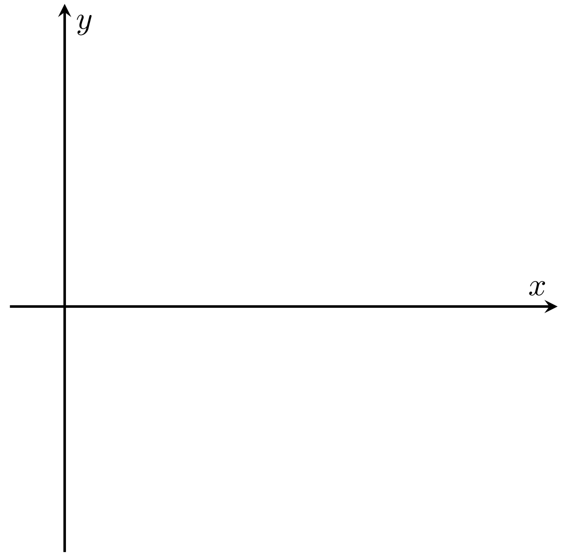
We can use the zero product property as above to find the ________________ of the graph.

To find the __________, we can use the symmetry of the parabola. The _______________________ passes through the __________, as well as exactly halfway between the ________________.

h is the ____________ of the zeros of the function, and k is the value of the function evaluated at h.



Example 5 Sketch a graph of $f(x) = (x-2)(x-10)$.



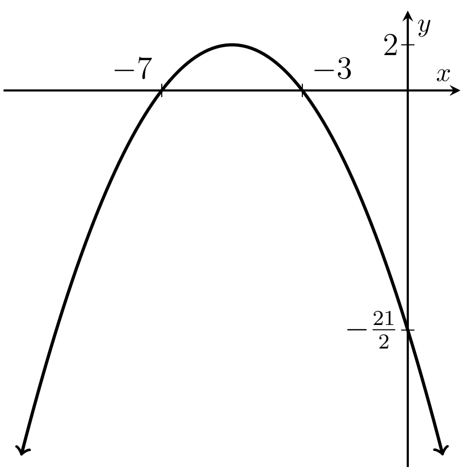
x-intercepts:

y-intercept:

vertex:

endpoints:

Example 6 Find the function g represented by the following graph.



x-intercepts:

y-intercept:

Example 7 Write $f(x) = (1-x)(x+6)$ in vertex form.

3.3 Review of Distributing and Factoring

The ____________________________ is one of the most important rules in algebra. Many of our results going forward are derived from it.

The Distributive Property

Example 1 Verify $8(7+5) = 8 \cdot 7 + 8 \cdot 5$

Example 2 Verify $3(20-6) = 3 \cdot 20 - 3 \cdot 6$

The process of changing $a(b+c)$ to $ab+ac$ is called ________________.

The reverse process is called ______________.

The _________________ can be used to ____________ the distributive property.

Distributing

To ______________ algebraically, multiply each _________ inside the parentheses by the __________ outside the parentheses.

Example 3 Distribute $3x(2x-4)$

Example 4 Distribute $-4y(7y^2+5)$

Example 5 Distribute $3x^2(x^4 - 2x^3 + 5x - 1)$

If there are _______ sets of parentheses, we need to _____________ over both. _________________ in the first set of parentheses is multiplied by ________________ in the second set of parentheses. After distributing, make sure you _________________________.

Example 6 Distribute $(x + 4)(x - 7)$

Example 7 Distribute $(2x + 3)(x + 6)$

Example 8 Distribute $(3x - 5)(x^3 + 2x^2 - 7)$

Factoring Using the Greatest Common Factor

If all the __________ in an expression have a __________ which is the same, that __________ is called a _____________________.

The ________________________________, or ________, is the largest possible _____________________ for the expression.

To factor, we can _________ every term by the ________, and write the result in __________________, with the ________ written in front. As the expression has been both __________ and _____________ by the ________, the result is equivalent.

This method of _____________ is the simplest and should be attempted _______. If this is done correctly, there will be no ______________________ remaining.

Example 9 Factor $9m^3 - 12m^2$

Example 10 Factor $12a^3b + 24a^2b^5 - 42a^4b^4$

Quadratics with Common Factors

We've already seen that ____________________ can be convenient for finding the zeros of a function. In certain circumstances, __________________________ can change a quadratic expression/function in ____________________ into ____________________.

Example 11 Solve $15x^2 + 10x = 0$

Example 12 Solve $2x^2 = 8x$

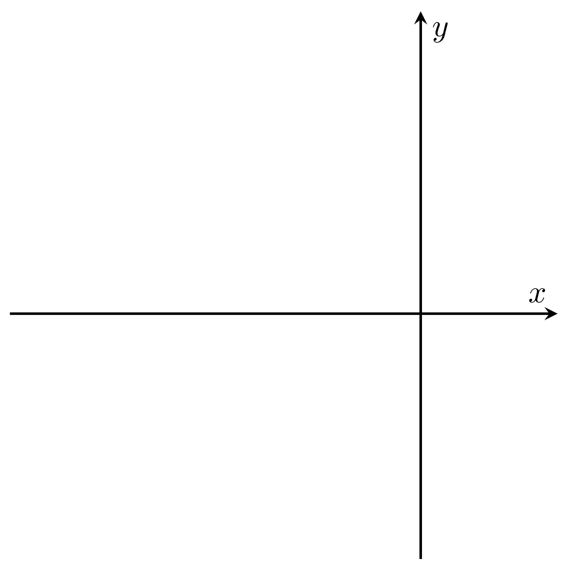
Example 13 Sketch a graph of $f(x) = -3x^2 - 15x$.

factor:

x-intercepts:

y-intercept:

vertex:



endpoints:

3.4 Special Quadratics

In the previous section, we factored select quadratics in standard form using the greatest common factor. The following rules will allow us to factor other special cases.

Theorem: Perfect Squares

Proof

Theorem: Differences of Squares

Proof

These rules can be used for _______________:

Example 1 Distribute $(x + 10)^2$

Example 2 Distribute $(2x + 7)(2x - 7)$

The rules can also be used for _____________:

Example 3 Factor $x^2 - 81$

Example 4 Factor $25x^2 - 30x + 9$

It is always a good idea to attempt to _____________________________ before factoring with any other method, including special quadratics:

Example 5 Factor $5x^2 + 20x + 20$

Example 6 Factor $63x^2 - 175$

As with all quadratic equations, equations in these forms can be solved using the _____________________________ if they are ____________:

Example 7 Solve $4x^2 + 196 = 56x$

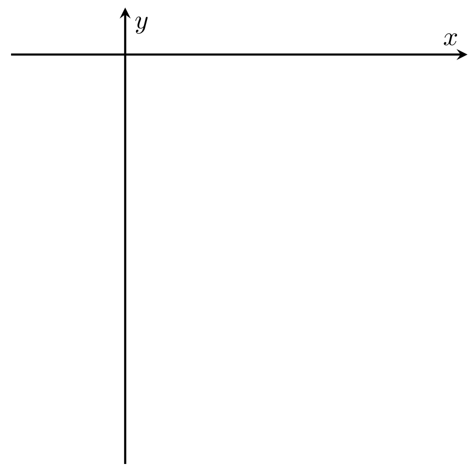
Example 8 Solve $12x^2 - 75 = 0$

Perfect Squares and Differences of Squares as Functions

Note that the ______________________ and _______________________________ rules are useful for converting these types of quadratic functions between their three forms:

	perfect square	difference of squares
standard form		
vertex form		
factored form		

Example 9 Sketch a graph of $f(x) = -2x^2 + 12x - 18$.



factor:

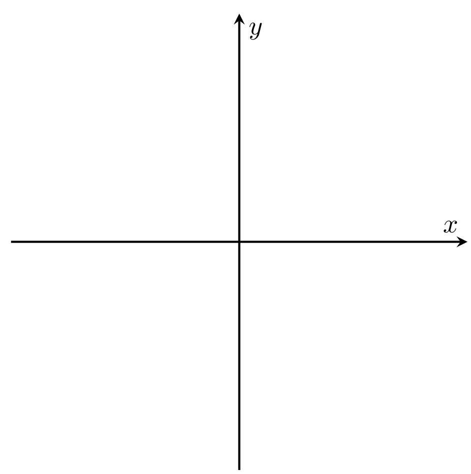
x-intercepts:

y-intercept:

vertex:

endpoints:

Example 10 Sketch a graph of $f(x) = 3x^2 - 12$.



factor:

x-intercepts:

y-intercept:

vertex:

endpoints:

Example 11 Write $g(x) = (x-5)^2 - 9$ in factored form.

Example 12 Write $h(x) = (x+7)^2 - 12$ in factored form.

Further Factoring Examples

While perfect squares and differences of squares are examples of ____________ expressions, they can also be used to factor certain other ______________[2].

Example 13 Factor $8x^4 - 18x^2$

Example 14 Solve $5x^3 + 60x^2 + 180x = 0$

Example 15 Factor $x^4 - 18x^2 + 81$

[2] We'll discuss polynomials in detail in a later chapter.

3.5 Factoring Quadratics in Standard Form

Recall that the ____________________ of a quadratic expression is

Factoring Monic Quadratics

A quadratic expression is called _________ if __________.

Theorem

If a monic quadratic expression $x^2 + bx + c$ has values p and q such that

and

then

Proof

Example 1 Factor $x^2 + 7x + 12$

Example 2 Factor $x^2 - 3x - 40$

Factoring Non-monic Quadratics

Often, a ______________ quadratic can be factored as if it were _________ by first factoring using the ________.

Example 3 Factor $6x^2 - 30x + 36$

Example 4 Solve $-4x^2 + 36x + 88$

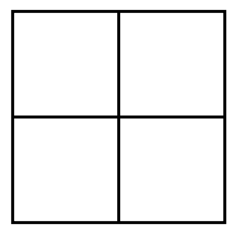
If this is not an option, then the following theorem can be used to help factor using the box method.

Theorem

In a 2×2 box using the box method, the ___________ of the values along each ___________ are the same.

Proof

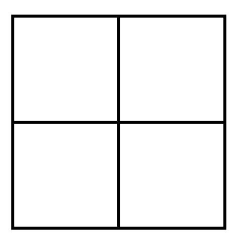
Consider the general expression ____________________, which is distributed using the box method.



Along the first diagonal: ____________________

Along the second diagonal: ____________________ ■

Example 5 Factor $5x^2 + 28x - 12$



The first diagonal contains _______ and ________.

The second diagonal has sum _______ and product _____________.

$\implies$ second diagonal is ________ and ________.

Finding common factors for each row and column gives

Example 6 Factor $12x^2 - 24x - 15$

Example 7 Factor $-12x^2 + 58x - 18$

Solving Equations by Factoring

Recall that a ____________ to an equation is a value which causes it to be ________. For quadratic equations, ______________ allows us to use the ________________________________ to find the solutions.

Example 8 Solve $x^2 + 15x + 36 = 0$

Example 9 Solve $x^2 + 5 = 8x + 14$

Example 10 Solve $4x^2 + 25x - 21 = 0$

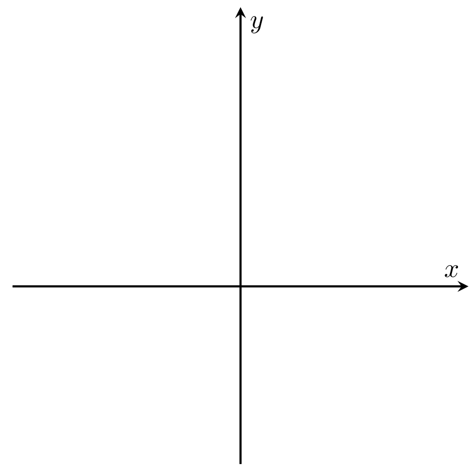
Example 11 Solve $20x^2 - 56x - 12 = 0$

Graphing Using Factoring

We've already graphed quadratic functions in ____________________. Using the same methods, we can graph quadratic functions in ____________________ if they can be ____________.

Example 12 Sketch a graph of $f(x) = x^2 + x - 2$.

factor:



x-intercepts:

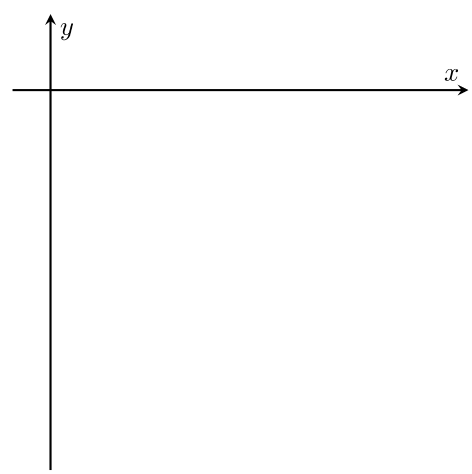
y-intercept:

vertex:

endpoints:

Example 13 Sketch a graph of $g(x) = -2x^2 + 9x - 9$.

factor:



x-intercepts:

y-intercept:

vertex:

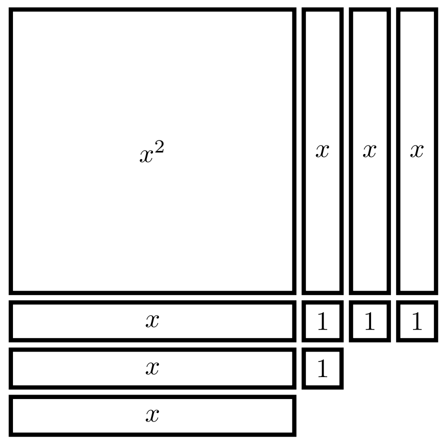
endpoints:

3.6 Completing the Square

While many quadratic expressions can be ____________ directly using the methods in the previous sections, most cannot. Instead, we use can a technique called ______________________________.

The goal is to rewrite the expression so that it contains a _____________________, which is then factored. The result is an expression in _________________. This makes it possible to ________ the related ____________, or _________ the related ____________.

The diagram to the right shows that $x^2 + 6x + 4$ is not a perfect square, but its square can be ______________ by adding and subtracting ____.



Example 1 Solve $x^2 + 6x + 4 = 0$ by completing the square.

Step 1: Identify the constant which completes the square.	
Step 2: Add and subtract to complete the perfect square.	
Step 3: Factor the perfect square to get vertex form.	
Step 4: Solve using the square root method.	

Example 2 Solve $x^2 - 10x + 7 = 0$

Example 3 Solve $x^2 + 2x - 5 = 0$

Example 4 Solve $x^2 + 3x + 1 = 0$

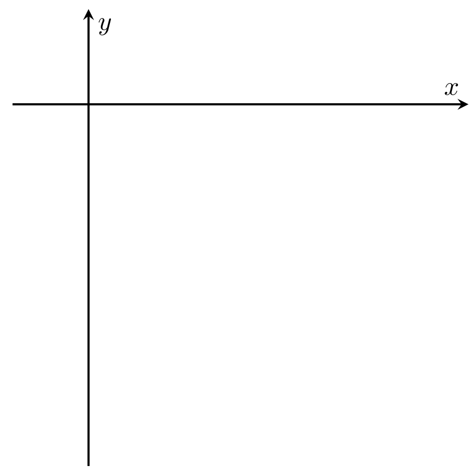
Example 5 Solve $4x^2 + 20x + 18 = 0$

Example 6 Write $f(x) = x^2 - 8x + 13$ in vertex form.

Example 7 Write $g(x) = -2x^2 - 20x - 59$ in vertex form.

Example 8 Sketch a graph of $f(x) = x^2 - 6x + 1$.

x-intercepts:



y-intercept:

vertex:

endpoints:

3.7 The Quadratic Formula

An alternative method to ______________________________ is using a ___________ to directly find the ____________ to a quadratic equation.

Theorem: The Quadratic Formula

A quadratic equation in standard form, _________________________, can be solved directly using the formula

Proof

$$ax^2 + bx + c = 0$$

$$x^2 + \quad x + \quad = 0 \qquad \text{divide both sides by } a \text{ (1)}$$

$$x^2 + \quad x + \quad - \quad + \quad = 0 \qquad \text{complete the square (2)}$$

$$\left(x + \quad\right)^2 - \quad = 0 \qquad \text{factor and simplify (3)}$$

$$\left(x + \quad\right)^2 = \qquad \text{isolate squared expression (4)}$$

$$x + \quad = \pm \qquad \text{take the square root (5)}$$

$$x = \qquad \text{finish solving for } x \text{ (6)}$$

■

The quantity ______________ is known as the ________________, denoted by _____, the upper case Greek letter ________. We can use it to state a simplified version of the quadratic formula.

Example 1 Solve $2x^2 + x - 28 = 0$

Example 2 Solve $3x^2 = 2x + 2$

Counting Real Solutions

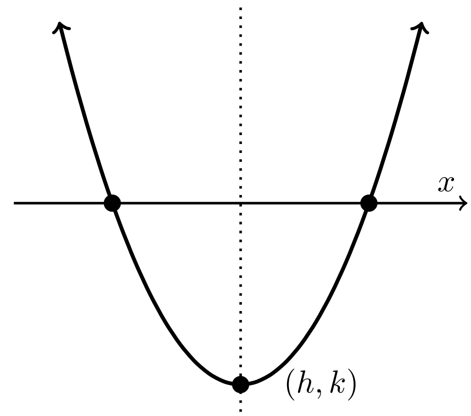
The ________ of the _________________ is particularly useful for finding the number of ___________________ to a quadratic equation. This also corresponds to the number of __________________ in the __________ of a quadratic function.

	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
solutions			
number of real solutions			
x-intercepts	$\xrightarrow{x}$	$\xrightarrow{x}$	$\xrightarrow{x}$

Graphing Quadratic Functions in Standard Form

Recall that the x-coordinate of the ___________, h, is the _____________ of the __________ of the function.

Since the __________ of the function are given by the _________________________, we get that their average is given by



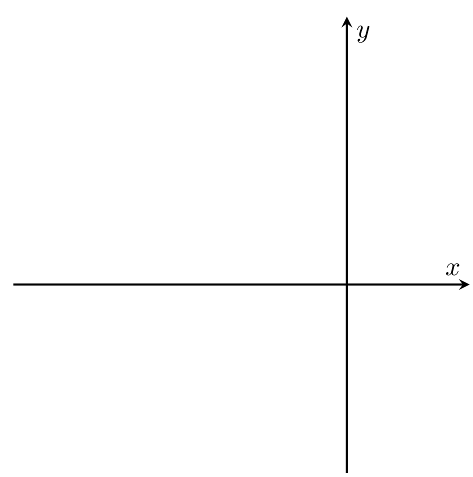
This formula holds even if there are not two real zeros.

This gives us the final tools we need for graphing quadratic functions in standard form.

shape of curve	
vertex	
x-intercepts	
y-intercept	
endpoints	

Example 3 Sketch a graph of $f(x) = -0.5x^2 - 3.2x + 5.8$, with x-intercepts to 2 decimal places.

x-intercepts:



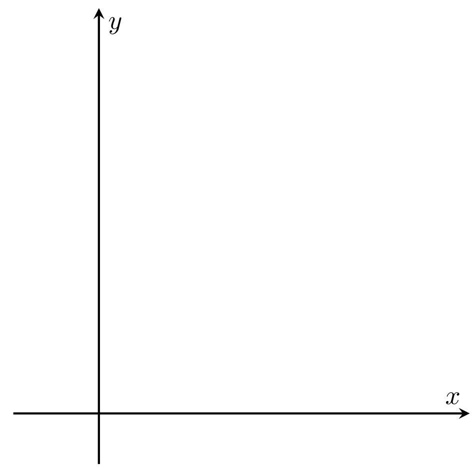
y-intercept:

vertex:

endpoints:

Example 4 Sketch a graph of $g : [0, 6) \to \mathbb{R}$, where $g(x) = 2x^2 - 8x + 11$

x-intercepts:



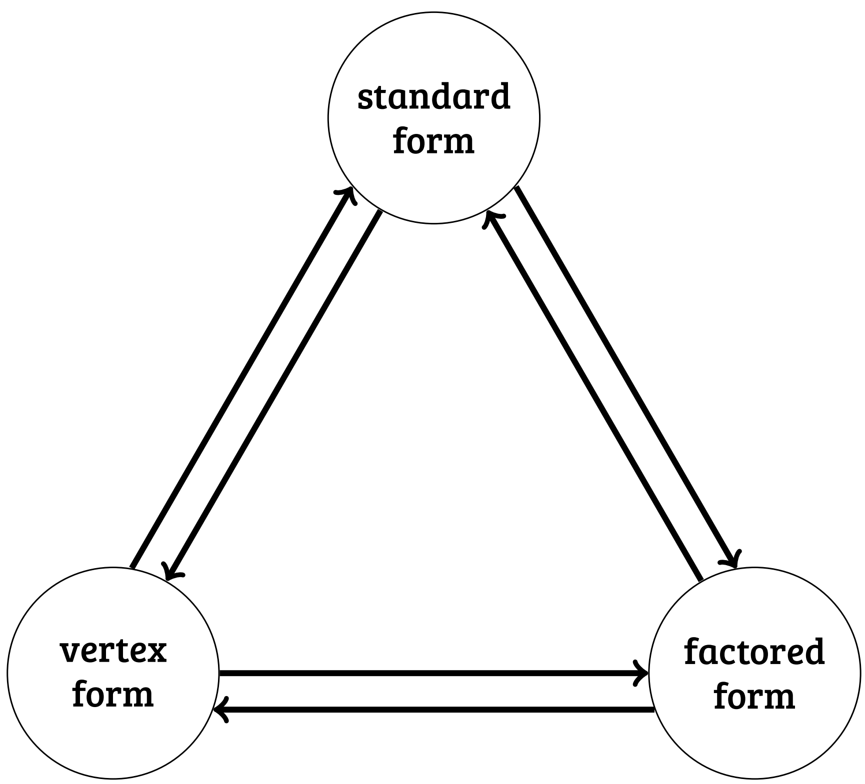
y-intercept:

vertex:

endpoints:

Converting Quadratics Between Forms

Throughout this chapter we've seen examples of converting between the three forms of quadratic functions. This diagram summarizes those methods.



In practice, if converting between vertex and factored forms, it's often easier to convert to standard form first.

Chapter 4

Further Quadratics

4.1 Complex Numbers

Recall that some ________________________ have ______________________, even if they are something simple, such as

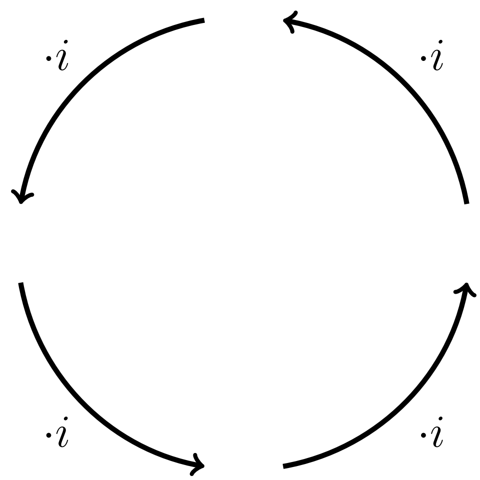
We can solve equations like this by introducing numbers outside the set of real numbers, known as ________________________.[1]

The __________________, denoted by ____, is a number defined as having the property

and is a solution to the equation above.

The ___________ of i follow a very particular pattern:

i^0		
i^1		
i^2		
i^3		
i^4		
i^5		
i^6		
i^7		
i^8		



Example 1 Evaluate each of the following.

i^{27} i^{394} i^{-23}

[1]Don't let the name fool you! Imaginary numbers may be abstract, but so are all numbers, and that doesn't mean they don't exist. Imaginary numbers have *many* applications in science and engineering. The mathematical terms *real* and *imaginary* are not entirely accurate, but they've been around for so long that we're stuck with them.

An ________________________ is any _________________ multiplied by ____.

A ______________________ is any number of the form __________ where a and b are real numbers. Note that if _________, the resulting complex number is real. Therefore, the real numbers are a __________ of the complex numbers.

Typed	Written	Name	Description
$\mathbb{C}$			The set containing all _______ and _____________ numbers, and their linear combinations.

For a given complex number, z, the _____________ is denoted by __________, and the ____________________ is denoted by __________.

Example 2 Find the real and imaginary parts of each of the following.

$z_1 = 3 + 7i$ $z_2 = -5 + 11i$ $z_3 = 9 - 13i$

Adding and Subtracting Complex Numbers

To add and subtract complex numbers, add and subtract the _______ and _____________ parts of the numbers independently. That is,

Example 3 Evaluate the following using z_1, z_2 and z_3 above.

$z_1 + z_2$ $z_2 + z_3$

$z_3 - z_1$ $z_1 - z_2$

Multiplying Complex Numbers

Complex numbers can be multiplied using the ______________________ as usual, which we can represent using the ______________. Don't forget to replace _____ with _______.

Example 4 Evaluate $(2+5i)(3-7i)$

Example 5 Evaluate $(-1-8i)(5-4i)$

Complex Conjugates

The ____________ of a complex number is the result of ____________ the _______ of the imaginary part of the number. The real part is ______________. ______________ is denoted by a __________________ over the number or variable.

Example 6 Find the conjugate of each of the following.

$z_1 = 3+7i$ $z_2 = -5+11i$ $z_3 = 9-13i$

Example 7 Multiply $z = 3-4i$ by its conjugate.

Dividing Complex Numbers

When we divide, the aim is to write the final result in the form __________, which takes a little more algebraic manipulation than the other operations.

This method relies on the property that the __________ of a complex number and its ____________ is a ________________.

1. Write the division as a __________.
2. __________ both the _____________ and _______________ by the ____________ of the _______________.
3. Evaluate each __________.
4. Simplify to the form __________.

Example 8 Simplify $\dfrac{2}{3+5i}$

Example 9 Simplify $\dfrac{3+4i}{5-2i}$

4.2 Quadratic Equations with Complex Solutions

Recall that when the ________________ of a quadratic equation, $\Delta = b^2 - 4ac$, is _____________, the equation has no _______ solutions. It turns out that these equations do indeed have solutions.

Theorem

Every quadratic equation $ax^2 + bx + c = 0$ has __________________ (when multiplicity[2] is considered), whose nature is determined by the ________________ $\Delta = b^2 - 4ac$:

1. If $\Delta > 0$, then there are _________________________________.
2. If $\Delta = 0$, then there is ______________________ with a multiplicity[2] of two.
3. If $\Delta < 0$, then there are _______________________________________.

Example 1 Solve each of the following equations with complex solutions.

$x^2 + 9 = 0$ $\qquad$ $x^2 + 75 = 0$ $\qquad$ $(x + 4)^2 + 36 = 0$

Generally, quadratic equations with complex solutions can be solved in the usual way using ______________________________ or ______________________________.

Example 2 Determine the nature of the solutions of $x^2 = 2x - 5$, then solve it.

[2] *Multiplicity* will be discussed in more detail in the *Polynomials* chapter.

Example 3 For each equation, determine the nature of the solutions. Verify by solving.

$-3x^2 + 4x - 2 = 0$

$4x^2 + 25 = 20x$

$3x^2 + 6x = 1$

4.3 Systems Involving Quadratic Equations

Quadratic-Linear Systems

Previously, we've worked with systems consisting of only ____________________. We now have the tools necessary to solve systems when ________________________ are included as well.

The meaning of a __________ to a quadratic-linear system is unchanged. A solution consists of values for __________________ which satisfy __________________ simultaneously (at the same time.) Because quadratics are involved, there may be _______, ______ or ______ real solutions.

As with ___________________, the goal is to algebraically manipulate the system so that all variables except one are _____________, resulting in a ___________________, which can be solved by the usual means.

Don't forget to _________________________________!

Example 1 Solve the system.

$$\begin{cases} y = x^2 + 6x - 33 & (1) \\ y = 3x - 5 & (2) \end{cases}$$

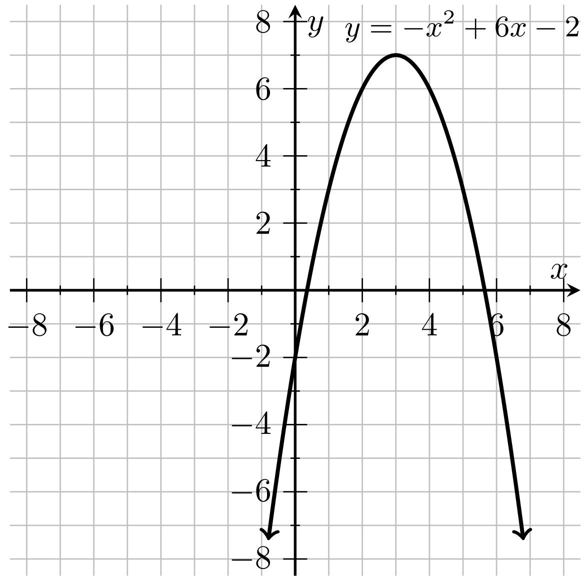
Example 2 Solve the system to 2 decimal places.

$$\begin{cases} x + 3y = 6 & (1) \\ \quad y = x^2 - 5 & (2) \end{cases}$$

Example 3 Graphically find the solutions to the system

$$\begin{cases} y = -x^2 + 6x - 2 \\ x + y = 4 \end{cases}$$

The curve for $y = -x^2 + 6x - 2$ is already plotted.



Example 4 Determine the number of real solutions of the system

$$\begin{cases} y = 5x + 11 \\ y = -x^2 + 2x + 8 \end{cases}$$

Example 5 Find k such that the system has exactly one solution.

$$\begin{cases} y = -x^2 + 4x - 4 \\ y = kx - 3 \end{cases}$$

Identifying Quadratics using Linear Systems

Suppose we know that a function f is quadratic, and that $f(3) = 5$. The function can be written in standard form as

which, by substituting $x = 3$ and $f(x) = 5$, becomes the equation

Is it possible to identify $f(x)$ from this equation?

Recall that a system in ____________________ requires ____________________ to be solvable.

Theorem

A ___________ function can be ___________ if it has ________________ at ________ points on the domain.

Example 6 Find the quadratic function f which satisfies $f(3) = 5$, $f(0) = -1$ and $f(4) = 15$.

4.4 Quadratic Regression

Recall that ________________ is the process of fitting a modeling function to a set of data in order to approximate the relationship between variables.

_____________________________ uses a ______________ function for the model. It is typical to use the _____________ form of the function. In practice, this means choosing values for _____, _____ and _____ so that _______________________________ fits the data as well as possible.

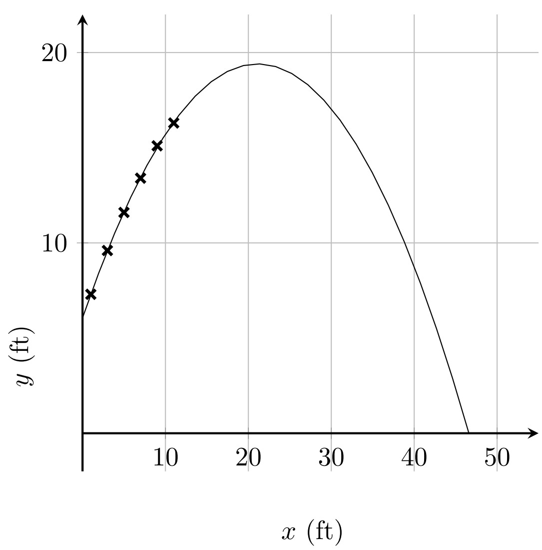
The ______________________________________ has the same meaning as for linear regression: it is a measure of how well the regression curve fits the data. For non-linear regression, _______ has no relation to _____.

Example 1 A camera captures the flight of a ball after it is thrown. The frames are analyzed, and the following data is recorded showing the horizontal distance, x, of the ball from where it was thrown versus its vertical height above the ground, y.

x (ft)	1.0	3.0	5.0	7.0	9.0	11.0
y (ft)	7.3	9.6	11.6	13.4	15.1	16.3

Use quadratic regression to model the flight of the ball.

Once technology is used to perform a ________________, it is usually simple to use the same technology to _______ the modeling function with the data, and perform further calculations related to the function.



Example 2 Comment on how well the model fits the data.

Example 3 Estimate the height of the ball after it has traveled 6.4 ft.

Example 4 Predict the maximum height of the ball, and the distance it will travel before hitting the ground.

Note that to answer the previous example, we had to use ________________, which may make the prediction unreliable. In this case, physics predicts that a "projectile" (such as the ball in the examples) has a parabolic path, which increases our confidence in our quadratic model, so the predictions seem sensible.

But suppose that someone catches the ball before it hits the ground. Then our prediction of the distance the ball will travel is incorrect. Always be careful using ________________, as additional information may be needed to accept or reject our predictions.

Chapter 5

Polynomials

5.1 Polynomial Concepts

A ______________ is an expression which, in standard form, can be written as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where

- n, and the following decreasing exponents, are ____________ greater than or equal to ________.
- $a_n, a_{n-1}, \ldots, a_0$ are ________________ (real numbers[1]).
- $a_n \neq 0$.

The largest _____________, n, is called the ___________ of the polynomial.

The _________ of a polynomial are the separate expressions of the form $a_i x^i$. The ______________ is the _______ of its _________.

Example 1 Write $P(x) = 9x^2 - 3x^3 - 11 + 12x^5 - 2x + 7x^2 + 5$ in standard form.

Naming Polynomials by Degree

degree	name	example
0		
1		
2		
3		
4		
5		

If the polynomial has a higher degree, it can be referred to as a ______________________________.

For example, $5x^9 - x^8 + 6x^7$ is a ______________________________.

[1]In general, mathematicians consider polynomials with coefficients of all sorts of number types. For us, they will always be real.

Naming Polynomials by Number of Terms

terms	name	example
1		
2		
3		

The name ______________ is a generalization of these names, with the prefix _________ meaning any number of terms fits the definition.

Example 2 $x^4 - 7x^2$ is a ___________ ___________.

Adding and Subtracting Polynomials

To add or subtract polynomials, add or subtract the ________________ of __________ with matching exponents.

Example 3 Add $3x^4 + 7x^3 - 9x^2 + 5$ and $-8x^4 + 5x^3 + 2x - 3$.

Example 4 Subtract $5x^4 - 3x^2 + 4x - 11$ and $x^4 - 7x^3 + 9x^2 - 6$.

Multiplying Polynomials

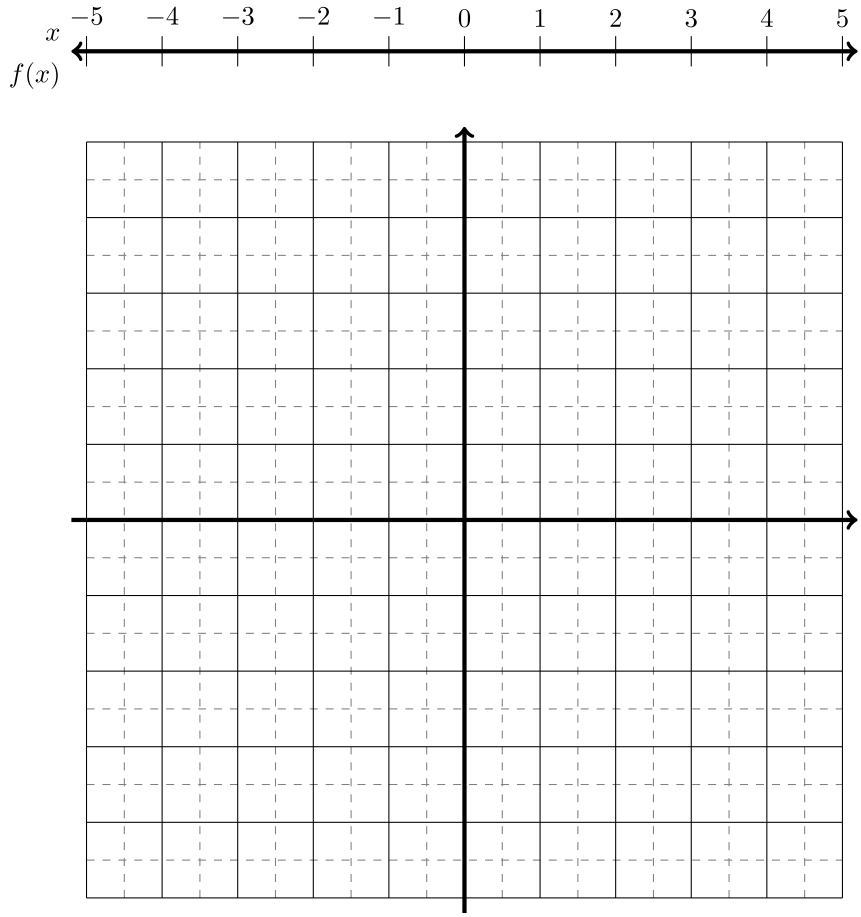
Polynomials are multiplied using the ___________________________, which was covered in Sec. 3.3.

Example 5 Distribute $(2x^2 - 7x)(x^5 + 3x^3 - 9x^2)$

5.2 Cubic Functions

Graphing polynomials becomes more difficult as their degree increases past two. An exception is functions resulting from ____________________ applied to the ___________________________.

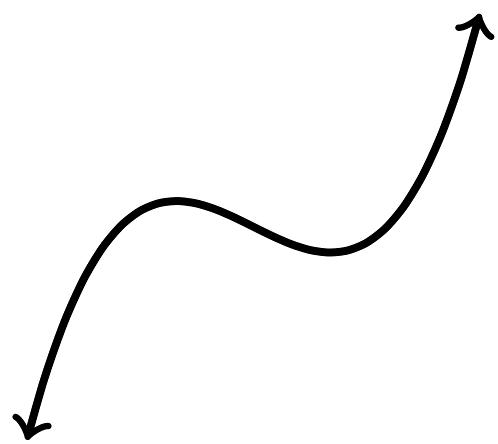
parent function
domain
range
relation type
x-intercept
y-intercept
point of inflection



The graphs of cubic functions have a point of _____________, which is a point where the _____________ changes direction.

In the case of the parent function $f(x) = x^3$, the curve changes from ___________________ to ________________ at $(0, 0)$.

Note that while the parent cubic function is ________________, this is not true of all cubic functions, including the one shown in the diagram here.

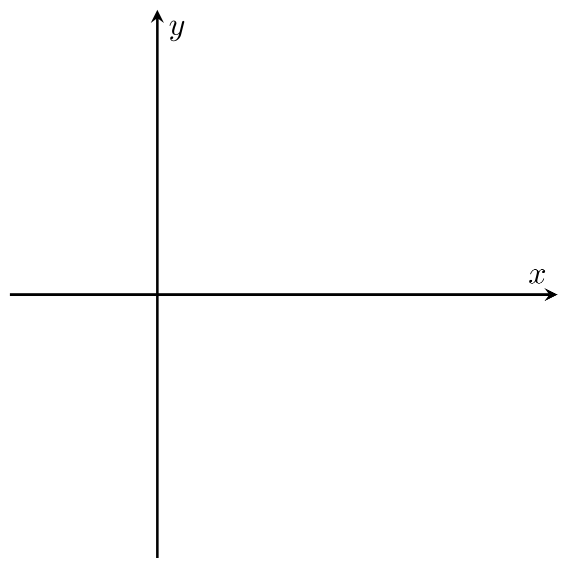


Graphing Cubic Functions Using Transformations

By applying ____________________ to the cubic parent function, we get the form ________________________________. Only a tiny subset of cubic functions can be written in this form. A sketch of this type of cubic function should include:

shape of curve	
point of inflection	
x-intercept	
y-intercept	
endpoints	

Example 1 Sketch $f(x) = \frac{1}{2}(x-3)^3 + 4$.



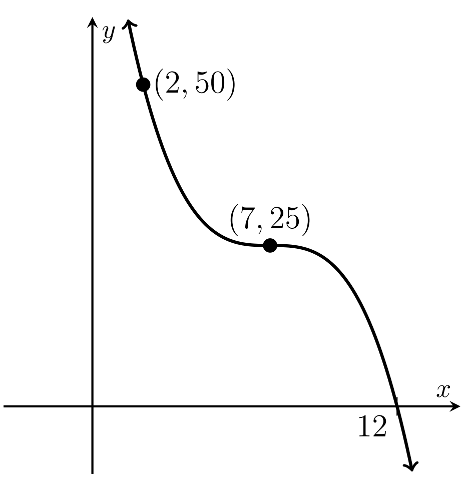
Orientation: Point of Inflection:

x-intercept:

y-intercept:

endpoints:

Example 2 Find the function g represented by the following graph.



Point of inflection:

Other point:

5.3 Special Cubics

Theorem: Perfect Cubes

Proof

Theorem: Sums and Differences of Cubes

Proof

As with the special quadratics in section 3.4, we can use these rules to quickly ______________ and __________ certain expressions.

Example 1 Distribute $(x-5)^3$

Example 2 Distribute $(x+4)(x^2-4x+16)$

Example 3 Distribute $(3x+7)^3$

Example 4 Factor x^3-1331

Example 5 Factor $x^3+12x^2+48x+64$

Example 6 Factor $729x^3-512$

Some expressions can be factored by combining these rules with others we've already learned.

Example 7 Factor $2x^8-1458x^2$

5.4 Polynomial Division

Recall from elementary school, before you learned decimals and fractions, that __________ of __________ results in a ______________ when the __________ isn't exact.

Example 1

$19 \div 7 =$ because

$35 \div 8 =$ because

$63 \div 11 =$ because

Note that the ______________ will always be smaller than the __________. The part of the result which is not the remainder is called the ___________.

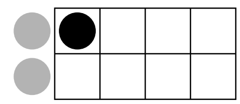
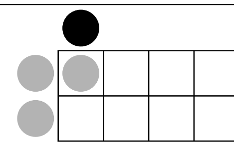
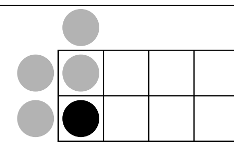
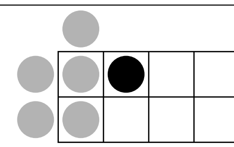
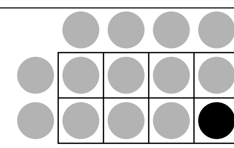
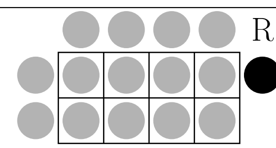
______________, as it turns out, are __________ in a manner very similar to ____________.[2]

Example 2 Verify that when $P(x) = x^4 - x^3 - 13x^2 + 28x - 9$ is divided by $x - 3$, the quotient is $Q(x) = x^3 + 2x^2 - 7x + 7$ and the remainder is 12.

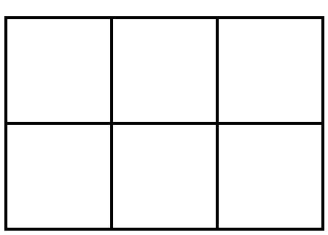
The goal of ______________________ is to find the ___________ and the ______________. There are several methods that can be used, but we will use a variation of the ________________ as we are already familiar with it.

[2]This isn't just a coincidence as it seems to be. Mathematicians actually consider the set of integers and the set of polynomials to have the same underlying algebraic structure.

In the final result, the __________ is placed along the left-hand side of the box grid, and the ____________ is placed along the top. The original ______________ is *mostly* contained within the grid, but won't fit perfectly if there is a _______________.

Step 1: Construct the box grid with the __________ along the ____________________.	
Step 2: Place the _______________ of the original polynomial in the _________________.	
Step 3: Remembering that the usual _________________ rules for the box method apply, complete the entry __________ the last entry.	
Step 4: Use _________________ to complete the column.	
Step 5: Complete the next cell in the ____________ so that its ____________ completes the _______________ in the original polynomial.	
Step 6: Repeat steps 3 to 5 until the _____________ is _____________.	
Step 6: ________________________ so that the ____________________ of the polynomial is complete.	

Example 3 Divide $P(x) = x^3 - 2x^2 - 21x + 7$ by $x + 4$.



Example 4 Divide $P(x) = 4x^3 - 6x^2 + 8$ by $x - 2$.

Example 5 Divide $x^4 + x^3 - 17x^2 - 42x - 66$ by $x^2 + 3x + 4$.

The Remainder Theorem

Recall that in integer division, the ______________ is always less than the __________.

A related idea for polynomials is described by the following theorem.

Theorem

In ______________________, if there is a ______________, its __________ is always less than the __________ of the _________.

If the _________ is ________, then the ______________ must be a ___________.

We can easily confirm that this is true for the examples above. In the particular case of a linear divisor, the following theorem is very important:

The Remainder Theorem

Suppose a ____________, $P(x)$, is _________
by a __________________, $x - a$.
Then the _____________ is equal to $P(a)$.

Proof

Let $Q(x)$ be the quotient, and let R be the remainder.

Example 6 Confirm the remainder from example 3, dividing $P(x) = x^3 - 2x^2 - 21x + 7$ by $x + 4$.

Example 7 Confirm the remainder from example 4, dividing $P(x) = 4x^3 - 6x^2 + 8$ by $x - 2$.

If the linear divisor is not __________, then we can use this updated version of the theorem.

Generalized Remainder Theorem

Suppose a _____________, $P(x)$, is _________ by a __________________
which equals _______ when $x = a$.
Then the _____________ is equal to $P(a)$.

Example 8 Suppose $P(x) = 2x^3 - x^2 + kx + 27$ is divided by $2x - 3$, and the remainder is 9. Find the value of k.

5.5 Factoring Polynomials

Suppose that a ____________ $P(x)$ is divided by a particular _________ $x - a$, and that the result is a __________ $Q(x)$ with ________________. This means we can write the statement

which means that $x - a$ is a _________ of $P(x)$.

The following is a special case of the ________________________, when there is ________________.

The Factor Theorem

$$x - a \text{ is a \_\_\_\_\_\_\_\_ of the \_\_\_\_\_\_\_\_\_\_\_\_ } P(x)$$
$$\text{iff (if and only if) } P(a) = 0.$$

This suggests a method we can use to _________ the polynomial $P(x)$:

Step 1: Find a value a for which $P(a) = 0$, which means $x - a$ is a _________.

Step 2: _________ $P(x)$ by $x - a$.

Step 3: Continue by ____________ the resulting ___________.

Example 1 Factor $P(x) = x^3 - 21x + 20$.

Example 2 Solve $2x^3 - 7x^2 - 8x + 28 = 0$

Example 3 Factor $P(x) = x^5 - 5x^4 - 25x^3 + 65x^2 + 84x$

5.6 Graphs of Polynomial Functions

Recall that a polynomial is a type of ______________. If it is treated as a function, then it is called a ________________________.

When _____________ the graphs of polynomial functions, we'll need to think about how the function ____________ in two different ways:

- _________, which means we only consider the immediate vicinity (close to) the ________ we're interested in; and
- __________, which means we consider the function over its entire __________.

Zeros, x-Intercepts and Multiplicity

For a polynomial function, as with all functions, the ________________ of its graph correspond to the _________ of the function, which are the ________ values which cause the __________ values to equal zero.

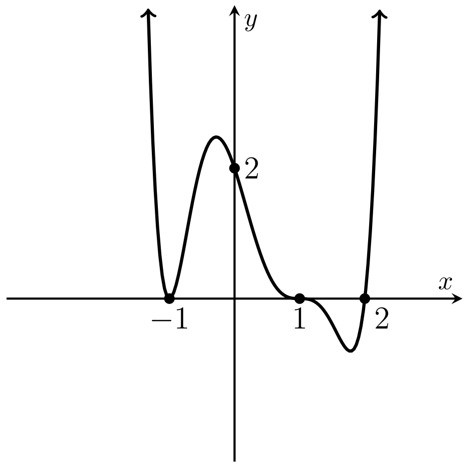
Example 1 Find the zeros of $f(x) = (x+1)^2(x-1)^3(x-2)$, and find the x-intercepts of its graph.

How many zeros are there in this example? If we count them, the simple answer is _________. If we're being more precise, we would say this is the number of __________ zeros.

But that's not the only way to count. Note that 1 is a ________ because $(x-1)$ is a __________ of the polynomial. But it's not a _________ just once, but _________ times. So we can say that 1 is a _____________________________. When we count the _________ with ______________, there are ______.

If a zero has _____________	1	2	3
the function behaves _________ like it is			
and the x-intercept is a			

The _______________ is found as in any function, at the point _____________.



Example 2 Identify the zeros and their multiplicity of the polynomial function f shown in the graph.

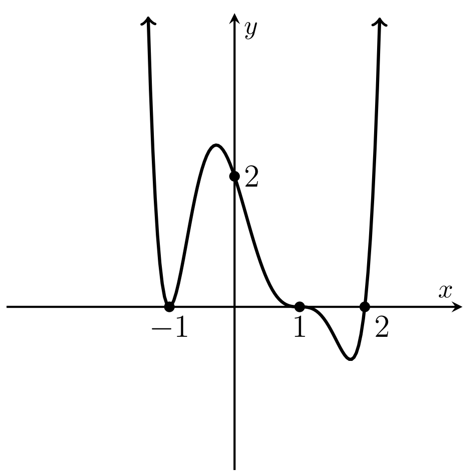
Positive and Negative Intervals

A ______________________ is an interval of the domain on which the value of the function is ____________, and its graph is __________ the x-axis.

A ________________________ is an interval of the domain on which the value of the function is ______________, and its graph is __________ the x-axis.

Keep in mind that a function's value is ________ at its zeros (by definition), and so is neither ____________ or ______________.

If a polynomial function changes ________, it will be at a ________, but not every ________ causes a change in ________.



Example 3 Identify the positive and negative intervals for the polynomial function f shown in the graph.

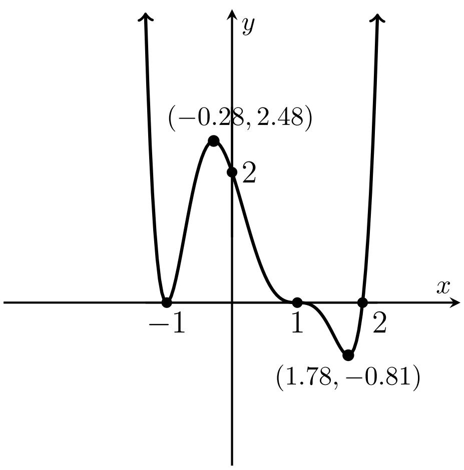
f is positive on the interval

f is negative on the interval

Minima and Maxima

A ____________________ of a function is a point at which the function has a *greater* value than any points nearby. A ___________________ of a function is a point at which the function has a *lesser* value than any nearby points nearby. For polynomial functions, these points occur at _____________.

The ______________________ of a function is the point at which the function has a greater value than at __________ other point in the domain. If it exists, it corresponds with either a ____________________ or an _____________. Similarly, the _____________________ has a value less than every other point and, if it exists, corresponds with a ___________________ or an _____________.



Example 4 Identify the (approximate) local and global maxima and minima for the polynomial function f shown in the graph.

f has a local maximum at

and has

f has local minima at

and has

Domain and Range

Polynomials can be evaluated for every real number, so the ___________ domain of a polynomial function is _____. If a graph shows ______________, however, the domain has been _______________.

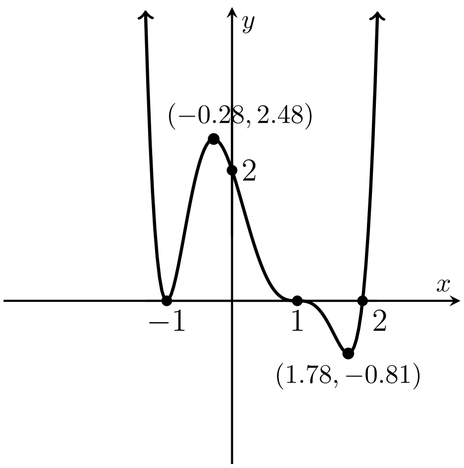
Knowing the global ______________ and/or _____________, if they exist, will typically allow us to find the __________.

Example 5 State the range of the function above.

Increasing and Decreasing

f is said to be ______________ if $f(x)$ increases as x increases, which implies ___________ slope.

f is said to be _______________ if $f(x)$ decreases as x increases, which implies ____________ slope.

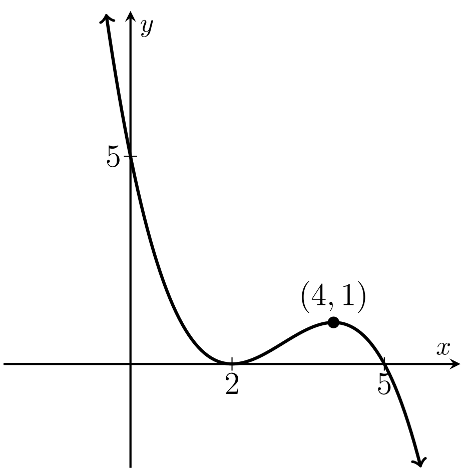


Example 6 Identify the increasing and decreasing intervals for the polynomial function f shown in the graph.

f is increasing on the interval

f is decreasing on the interval

Example 7 Find a polynomial function g to fit the following graph.



Chapter 6

Rational Expressions and Functions

6.1 Simplifying Rational Expressions

Recall that a ____________________ is a number which can be written in the form of a fraction, where the ______________ and ________________ are both ____________.

Examples	**Non-examples**

Similarly, a ________________________ is an expression which can be written in the form of a fraction, where the ______________ and ________________ are both _______________.

Examples	**Non-examples**

Also recall that any _________ (with a key exception) divided by ________ is equal to ____. You should be familiar with using this property to ______________________.

Examples $\dfrac{9}{6}$ $\dfrac{50}{60}$

We can use the same property to __________________________________.

Example 1 Simplify $\dfrac{(x+2)(x-5)}{x-5}$

However, if the value being divided by itself is ________, then the expression cannot be ____________ like this. Our example has this issue when __________. If this is the case, the original expression and the simplified version are not ______________.

The solution to this problem is to ___________________ from our simplification. We call this an ___________________, and we write the result as

Example 2 Simplify:

$$\frac{12x^3}{3x}$$

Example 3 Simplify:

$$\frac{(x-5)(x+3)(x-6)}{(x-6)(x+3)(x+5)}$$

Example 4 Simplify:

$$\frac{4-x^2}{x^2+x-6}$$

Example 5 Simplify:

$$\frac{x^3+125}{x^3+15x^2+75x+125}$$

An Error to Avoid

Remember that only __________ can be eliminated by dividing, not __________. With an expression like the one in example 4, a common error is to do the following.

Don't do this: $$\frac{\cancel{x^2}+5x+6}{\cancel{x^2}+x-6}=\frac{5x+6}{x-6}$$

This is because the __________ operation of division is ________________, not __________ or ______________.

Multiplying and Dividing Rational Expressions

Recall that fractions can be ____________ by multiplying the _______________ and multiplying the ________________.

Example $$\frac{3}{5}\cdot\frac{11}{6}$$

Also, recall that __________ by a fraction is the same as multiplying by its _____________.

Example $$\frac{4}{7}\div\frac{8}{9}$$

Note that in these examples, some simplifying could have been done at the start.

The same methods can be used to ___________ and _________ rational expressions. It is always a good idea to __________ and ___________ whenever possible.

Example 6 Simplify:

$$\frac{x^2-2x-8}{x+3}\cdot\frac{x+3}{x^2+4x-32}$$

Example 7 Simplify:

$$\frac{x^2+12x+35}{3x^2+x-10}\cdot\frac{x^2+9x+14}{x+5}$$

Example 8 Simplify:

$$\frac{x^2+7x-30}{x-4}\div(x^2+6x-40)$$

Example 9 Simplify:

$$\frac{x^2+7x+10}{x^2-x-6}\div\frac{x^2+6x+5}{x^2+x-12}$$

In the last example, there's an extra ____________________ at -4. The factor __________ is not eliminated, but it is originally in a ________________. If $x=-4$, the original expression is ______________.

6.2 Adding and Subtracting Rational Expressions

Recall that ______________ can be __________ or ________________ if they have the same __________________.

Examples

$\frac{2}{5}+\frac{7}{10}$

$\frac{3}{4}-\frac{1}{6}$

Similarly, ____________ expressions can be __________ or ________________ if they have the same __________________.

Example 1 Simplify:

$$\frac{x^2+8x}{x^2+7x+12}-\frac{10x+24}{x^2+7x+12}$$

Example 2 Simplify:

$$\frac{x-12}{x-3}+\frac{4x+15}{x^2-3x}$$

Finding the Lowest Common Multiple

The _________________________________ of two (or more) expressions is the _____________ expression which is a ____________ of each given expression.

To find the _________, find the simpliest ______________ for each expression so that each has the same ____________, which is the _________.

Example 3 Find the lowest common multiple of $5x$, $10x^2y$ and $15y^3$.

Example 4 Find the lowest common multiple of $(x-6)^2$ and $(x-6)(x+8)$.

Example 5 Find the lowest common multiple of $x(x-2)$ and $(x-2)(x+5)$.

Example 6 Find the lowest common multiple of $x^2+9x+20$ and $x^2-2x-35$.

Adding or Subtracting with Different Denominators

If the __________________ are different, we look to find the _________ of the __________________, and make that the ____________________________.

It is best practice to ____________ and __________ the resulting _______________, in case the expression can simplify further.

Example 7 Simplify:

$$\frac{x}{x+1} - \frac{4}{x+4}$$

Example 8 Simplify:

$$\frac{5}{x^2 + 9x + 14} + \frac{x}{x^2 + 6x + 8}$$

Example 9 Simplify:

$$\frac{x}{x^2 - x - 6} - \frac{9}{x^2 + 9x - 36}$$

6.3 Complex Fractions

We've already learned that a rational expression is a fraction with polynomials for the numerator and denominator.

If the numerator and denominator of a fraction are ___________________________ themselves, the fraction is a _______________________. These expressions are complicated, as their name suggests[1], so it is desirable to ___________ them as much as possible.

If the numerator and denominator each contain only a ____________________, then the complex fraction is simply just ___________ of two rational expressions, written in a different form. This means they can be treated in the exact same way, by _______________ by the ______________ of the __________.

Example 1 Simplify:

$$\frac{\frac{x+3}{x}}{\frac{x}{x+1}}$$

If a complex fraction contains a ________ or _______________ of rational expressions, then there are a couple of options to ___________ them.

Method 1: Multiply by Denominators

In this method, we eliminate the __________________ of the smaller fractions by _______________ everything by their ___________.

Example 2 Simplify:

$$\frac{\frac{1}{x}+\frac{2}{x+5}}{\frac{x}{x+5}}$$

[1]The name "complex fractions" does not imply they are related to complex numbers. If you want a less confusing name, you could call them "nested fractions."

Method 2: Adding and Subtracting First

In this method, we simplify the ______________ and/or the ________________ as we would for any expression with addition or subtraction. Then treat the result as __________.

Example 3 Simplify:

$$\frac{\frac{1}{x}+\frac{2}{x+5}}{\frac{x}{x+5}}$$

Example 4 Simplify:

Using Method 1:

$$\frac{\frac{x-7}{x^2-9}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{x^2-9}}$$

Using Method 2:

$$\frac{\frac{x-7}{x^2-9}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{x^2-9}}$$

6.4 Rational Equations

An equation which consists of ________________________ is called a ____________________. As with any equation, _________ means finding the values for the __________ which make the equation _______.

To simplify the equation, we can eliminate the ________________ by multiplying the entire equation by their __________. This reduces the equation to a _____________ equation, which is frequently a ____________ equation. We can then use our typical methods to finish solving.

Example 1 Solve $\dfrac{x+2}{x-2} - \dfrac{x+9}{x} = 1$

We can check that both solutions are _______ by _______________ them into the original equation.

In this case, both of the solutions __________ the equation. This is not always true, which is why we need to check the solutions.

For rational equations, it is possible to obtain ________________ solutions. ________________ solutions, which are not actually solutions, appear when the equation is solved, but are ________________ with the original equation.

Example 2 Solve $\dfrac{x-3}{x+3}+\dfrac{2}{x-2}=\dfrac{5x}{x^2+x-6}$

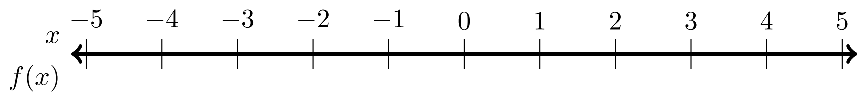
Checking the solutions:

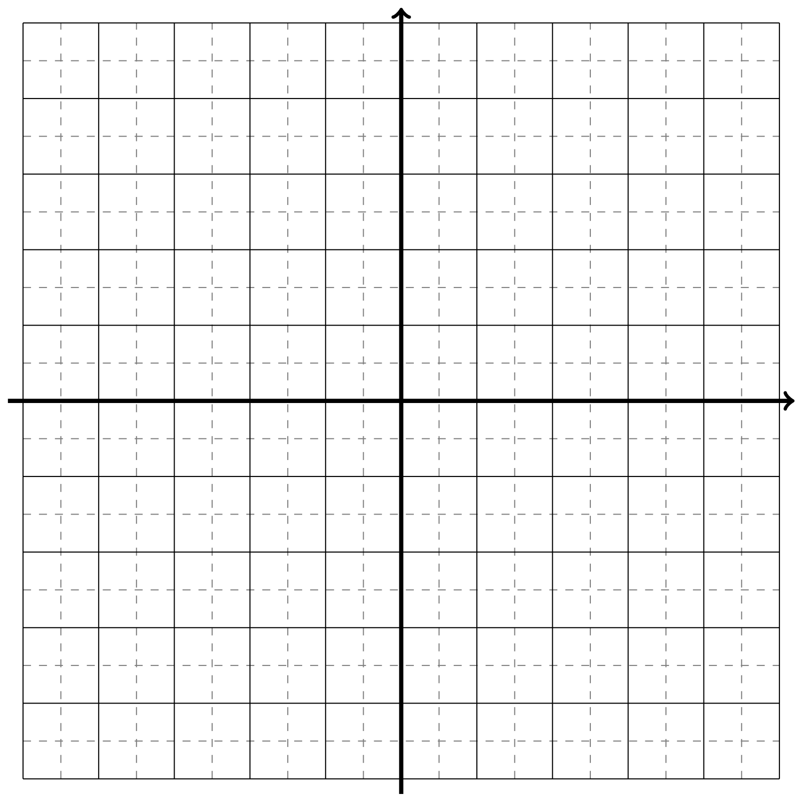
Because extraneous solutions can arise from rational equations, you must ____________________ your solutions with the original equation.

6.5 Simple Rational Functions

The simplest non-trivial rational function is the ________________________.

parent function
domain
range
relation type
horizontal asymptote
vertical asymptote
shape





An ______________ is a line which a function's curve continues to get _________ to, without ever _____________ it.

This function has a __________________________ at _________, because as x ______________ towards $+\infty$ or ______________ towards $-\infty$, $f(x)$ continues to get _________ to zero.

$$\text{As } x \to \pm\infty,\ f(x) \to 0$$

The function also has a ________________________ at _________, because as x gets _________ to zero, $f(x)$ continues to ____________ $+\infty$ or _____________ to $-\infty$.

$$\text{As } x \to 0,\ f(x) \to \pm\infty$$

Transformations of the Reciprocal Function

By applying ____________________ to $y = \dfrac{1}{x}$, we arrive at the ____________________

A sketch of this type of function should include:

shape of curve	
x-intercept	
y-intercept	
vertical asymptote	
horizontal asymptote	
endpoints	

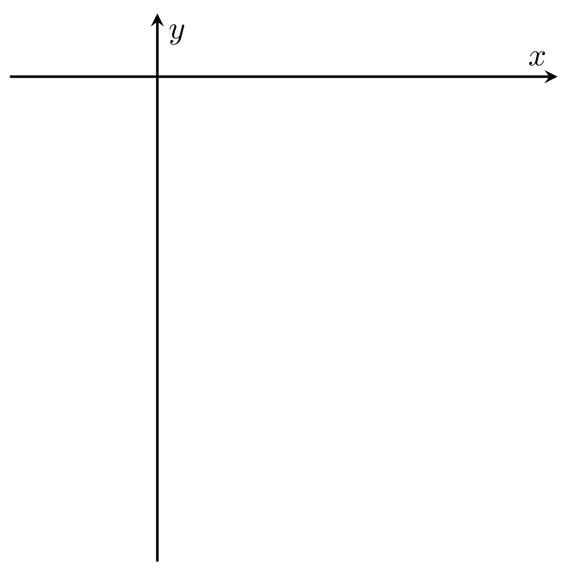
The points one unit left and right of the vertical asymptote are useful for guiding the overall shape of the graph.

Example 1 Sketch a graph of $f(x) = \dfrac{-1}{x-3} - 5$, and state its domain and range in three forms.

Orientation:

Asymptotes:

x-intercept:

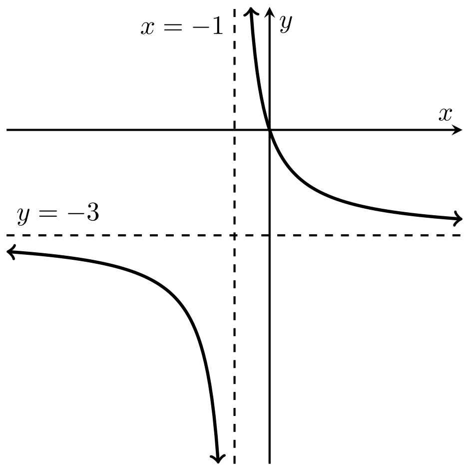


y-intercept:

Other points:

Domain: Range:

Example 2 Find the function g represented by the following graph.



Inverses of Simple Rational Functions

Functions of the form $y = \frac{A}{x-h} + k$ are ________________, which means they each have an ______________________. It turns out that the ________________________ have the ________________. Finding _____________ follows the same process we used in section 2.2.

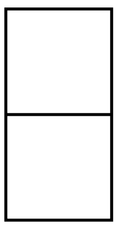
Example 3 Find the inverse of $f(x) = \dfrac{1}{x-2} + 7$. State the domain and range of f, and the domain and range of f^{-1}.

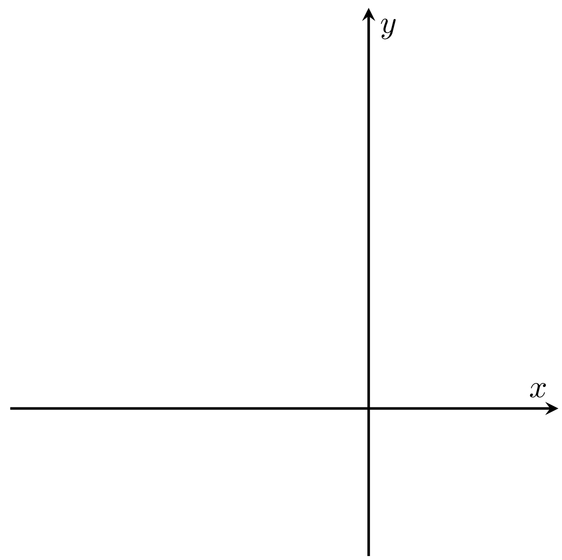
Linear Rational Functions

A rational function whose numerator and denominator are both __________ has a ________________ for its graph, just like $y = \frac{A}{x-h} + k$, though determining its characteristics is more difficult. To handle these functions, we can use __________________________ (section 5.4) to convert their form.

Example 4 Write $f(x) = \dfrac{3x+8}{x+2}$ in the form $y = \frac{A}{x-h} + k$, and sketch its graph.

You can use the known values $f(0) = 4$ and $f(-\frac{8}{3}) = 0$.



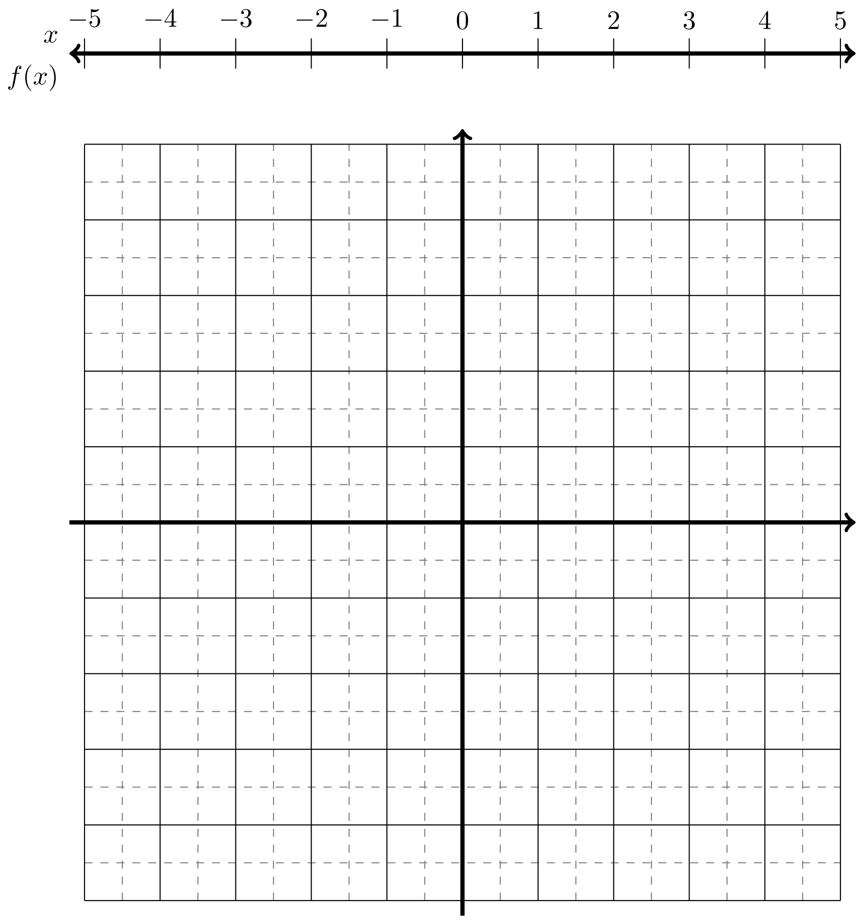


Example 5 Write $g(x) = \dfrac{-2}{x-6} + 7$ in the form $y = \dfrac{ax+b}{cx+d}$.

6.6 Functions with Quadratic Denominators

Transformations of x^{-2}

parent function
domain
range
relation type
horizontal asymptote
vertical asymptote
shape



This parent function is similar to the ______________ function. It has the same ___________, and its graph has the same ________________. However, because x is ____________, the output values are all ____________, which changes the __________.

Note that the shape of a curve is not a ______________, but is a slightly different shape called a ____________.

By applying _____________________, we arrive at the ___________________

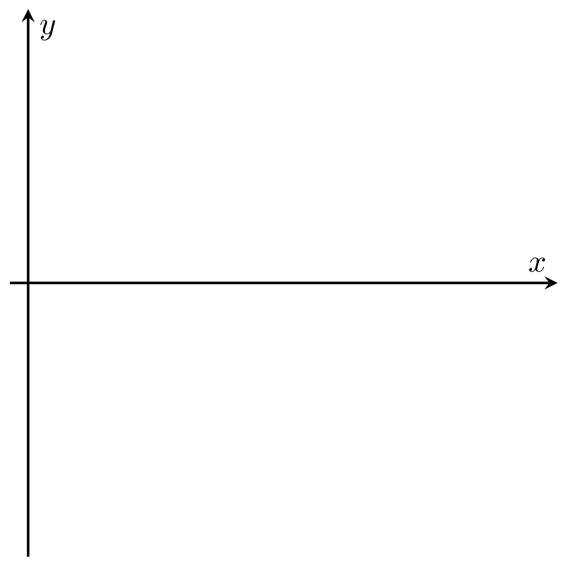
A sketch of this type of function should include:

shape of curve	
x-intercepts	
y-intercept	
vertical asymptote	
horizontal asymptote	
endpoints	

Example 1 Sketch a graph of $f(x) = \dfrac{9}{(x-7)^2} - 4$.

Asymptotes:

x-intercept:



y-intercept:

Other points:

Example 2 Find the rule for a rational function f with an implied domain of $(-\infty, -2) \cup (-2, \infty)$ and a range of $(-\infty, 8)$. The function does not represent a stretch or compression applied to the parent function.

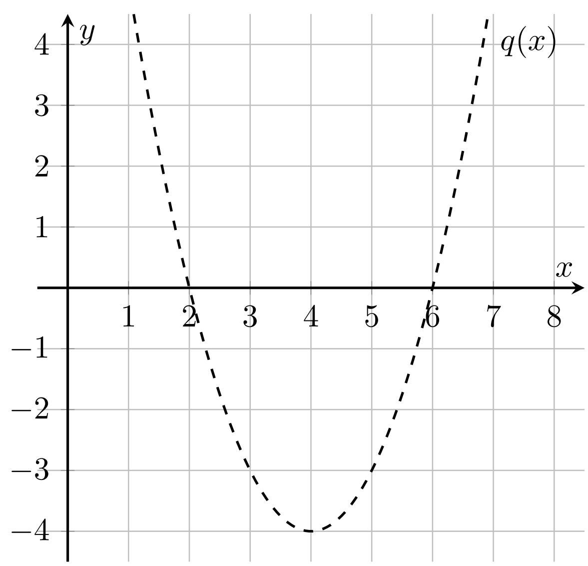
Reciprocals of Quadratic Functions

Functions of the form $f(x) = \dfrac{1}{q(x)}$, where $q(x)$ is a ____________ function, can be graphed by examining the behavior of $q(x)$.

If ____________ function $q(x)$...	...then its _____________ $f(x) = \dfrac{1}{q(x)}$...
has a zero at x	
has a local minimum (h, k)	
has a local maximum (h, k)	
approaches $\pm\infty$	
is positive	
is negative	
equals ± 1	

Example 3 Draw the graph of $f(x) = \dfrac{1}{x^2 - 8x + 12}$. The graph of $q(x) = x^2 - 8x + 12$ is already shown.

Asymptotes:



y-intercept:

Vertex:

Note that you won't typically be given the parabola for the quadratic in practice questions. It's still a good idea to draw it first before attempting to draw its reciprocal.

Example 4 Sketch a graph of $f(x) = \dfrac{2}{x^2 - 4x + 5}$.

Rewrite $f(x)$ in the form $\dfrac{1}{q(x)}$:

Properties of $q(x)$:

Zeros:

y-intercept:

Vertex:

Equals ± 1:

Properties of $f(x)$:

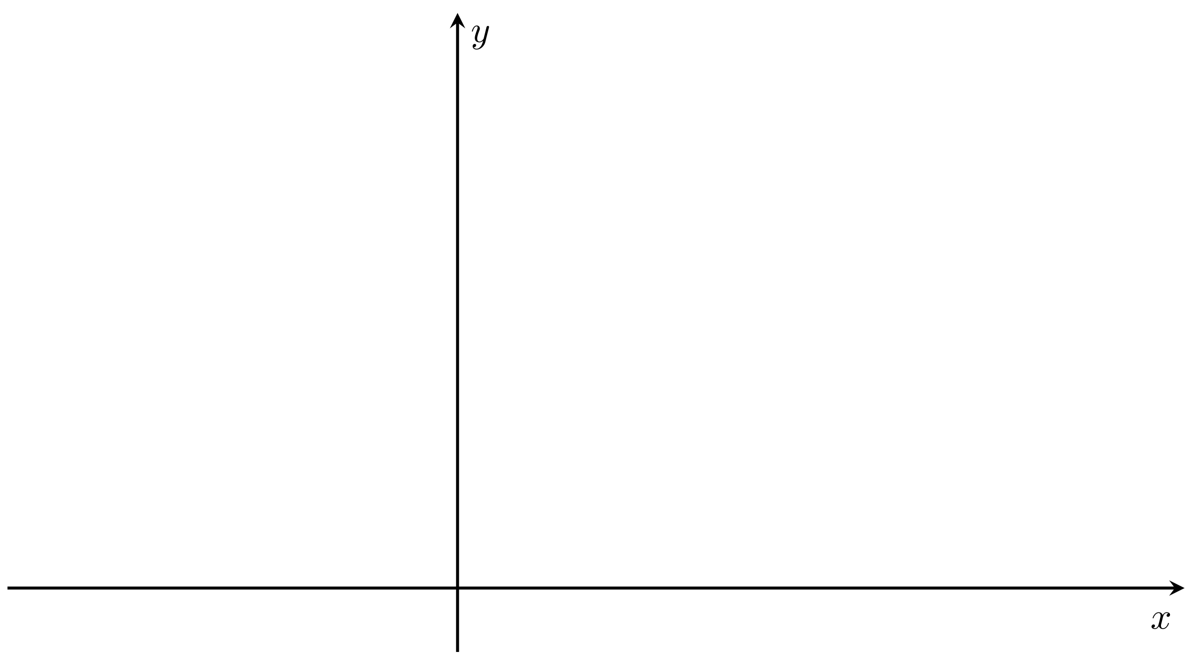
Vertical Asymptotes:

y-intercept:

Vertex:

Equals ± 1:

Horizontal Asymptote:



Chapter 7

Radicals and Rational Exponents

7.1 Radical Expression Concepts

Recall that the ____________ of x is the value y such that $y^n = x$, which we write as

- The symbol _______ is the _________ symbol.
- The small number written over the radical ____ is called the ________.
  (Don't mix this up with a **coefficient** written in front of the radical.)
- The value ____ under the _________ is called the ___________.

The 2nd root is called the _______________, and is usually written without the ________.

The 3rd root is called the _____________.

Example 1

$\sqrt{81} =$ because

$\sqrt[3]{125} =$ because

$\sqrt[5]{32} =$ because

Simplifying Radicals

It is conventional to write radical expressions with the smallest possible value in the ___________.
This is done by identifying a _________ which has a __________ nth root.

Example 2 Simplify the following.

$\sqrt{72}$ $\sqrt[3]{108}$ $\sqrt[6]{128}$

The same principle can be used when there are ____________ in the ___________.

Example 3 Simplify the following.

$\sqrt{75x^7}$ $\sqrt[3]{48x^5}$ $\sqrt[4]{81xy^5}$

Adding and Subtracting Radicals

Radical terms with the same ____________ and _________ can be added or subtracted by adding or subtracting their ________________, just as _______________ are simplified.

Some radicals may need to be _____________ first.

Example 4 Simplify the following.

$$9\sqrt{6} - 7\sqrt{3} + \sqrt{6} + 4\sqrt{3}$$

Example 5 Simplify the following.

$$2\sqrt{45} + 3\sqrt{50} - 6\sqrt{8} + 4\sqrt{20}$$

Multiplying Radicals

Radicals with the same index can be multiplied by multiplying their ______________. If each radical has a _______________, these are multiplied together.

Example 6 Simplify the following.

$$3\sqrt{10} \cdot 7\sqrt{2}$$

$$2\sqrt{7} \cdot 5\sqrt{14}$$

If binomial expressions are being multiplied, then we can use the ___________________________.

Example 7 Simplify the following.

$$3\sqrt{2}(\sqrt{5} + 4\sqrt{2})$$

Example 8 Simplify the following.

$$(2 + \sqrt{5})(7 - 6\sqrt{5})$$

Dividing Radicals

When dividing radicals, it is considered good practice to ensure the ________________ is __________, in a process called ____________________________________.

If the ________________ has _______ term, we can multiply by an appropriate radical to make it __________. In the case of a square root, we can use the _______________________.

Example 9 Rationalize the denominators.

$$\frac{3\sqrt{7}}{5\sqrt{3}}$$

$$\frac{4\sqrt[3]{6}}{3\sqrt[3]{2}}$$

If the ________________ has _______ terms involving square roots (but not higher roots), we can make it __________ by multiplying by its _____________, following the same process we used for dividing complex numbers in section 4.1.

Example 10 Rationalize the denominator.

$$\frac{6\sqrt{2}+7\sqrt{3}}{3\sqrt{2}+5\sqrt{3}}$$

7.2 Rational Exponents

Review of Exponents

An ____________ is used to indicate repeated ______________ of a number called the ________.

where n is the ____________ and a is the ________.

Exponent Product Rule

Exponent Quotient Rule

Exponent Power Rule

Negative Exponent Rule

Base Product Rule

Base Quotient Rule

Special Value Zero

Special Value One

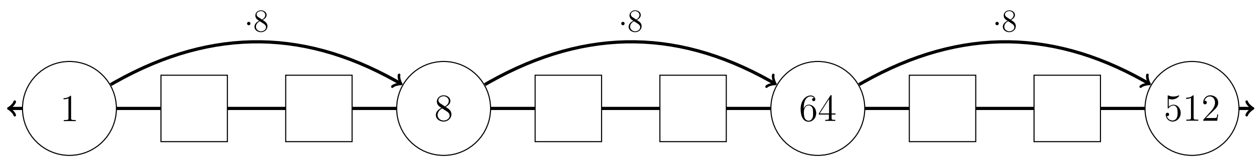
Rational Exponents

When an exponent is a ____________, it is known as a ________________________. We can use the ______________________________ to help evaluate them.

Example 1 Evaluate the following.

$36^{1/2}$ $\qquad$ $81^{3/4}$ $\qquad$ $8^{7/3}$

Let's take a closer look at the last example and consider what $8^{7/3}$ actually means. Recall that an ______________ indicates how many times the _________ is multiplied by itself. From the diagram it's simple to see that, for instance, multiplying by 8 __________ times results in $8^3 =$



But what does it mean to multiply 8 seven-thirds times, since it is not an ____________? Consider that multiplying by 8 once is the same as multiplying by _____ three times. It follows that multiplying by 8 "one-third times" is equivalent to multiplying by _____ _________.

Finally, this means that multiplying by 8 seven-thirds times is the same as multiplying by _____ __________ times, and that $8^{7/3} =$

Roots and Exponents

Consider the following:

$\sqrt{36}$ $\qquad$ $\left(\sqrt[4]{81}\right)^3$ $\qquad$ $\left(\sqrt[3]{8}\right)^7$

Notice that we're performing the ________________________ as the example above, with the _________ of the root taking the place of the _________________ of the exponent. This is because radicals and rational exponents are _______________.

Theorem: Roots and Rational Exponents

Proof

Example 2 Write the following in exponent form.

$\sqrt[5]{11}$ $\sqrt{6^9}$ $(\sqrt[4]{21})^{13}$

Example 3 Write the following in radical form.

$7^{1/6}$ $31^{5/3}$ $10^{11/2}$

Example 4 Evaluate the following.

$25^{1/2}$ $32^{3/5}$ $343^{4/3}$

Example 5 Simplify the following.

$\left(\sqrt[4]{x}\right)^{12}$ $\sqrt[6]{x^3}$ $\sqrt[12]{16}$

7.3 Square Root Equations

Recall that to solve rational equations, we converted them into polynomial equations, which we then solved using the usual methods. For equations with ________________ we can take a similar approach.

Like rational equations, equations with ________________ can have ________________________,

so each solution needs to be checked against the ____________________.

Example 1 Solve $x = \sqrt{7x + 15} - 1$.

Step 1: Rearrange the equation to isolate the _______________.	
Step 2: Eliminate the _______________ by ____________ both sides.	
Step 3: Solve the resulting equation.	
Step 4: Check for _______________ solutions.	
Step 5: State the _______ solutions.	

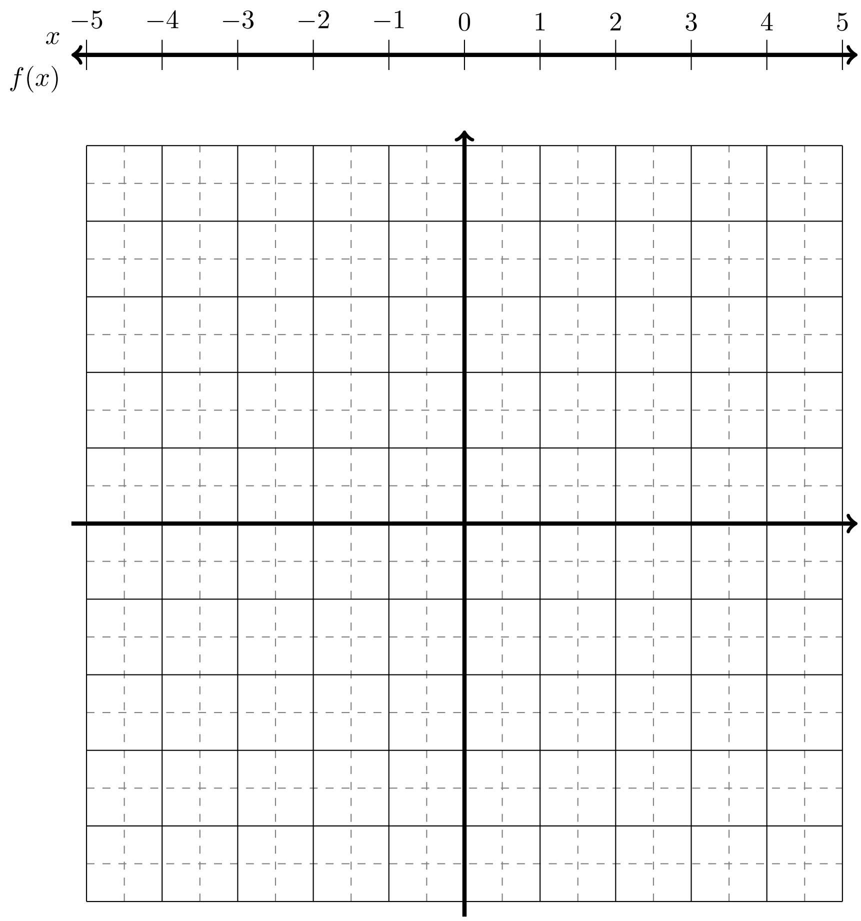
Equations with ___________ square roots are more challenging to solve, and require _____________ more than once, as only one root can by ___________ at a time. Care is needed to apply the _____________________ rule appropriately.

Example 2 Solve $\sqrt{x+4}+3=\sqrt{7x+1}$.

7.4 Square Root Functions

Functions which contain a __________ can be called ______________________. For this class, we will consider _________________ and ______________ functions.[1]

parent function
domain
range
relation type
x-intercept
y-intercept
endpoint



As the inverse of _____________ functions, square root functions have ______________ for their curves, though facing a different direction. Half of the ____________ is missing; if the bottom half was present, it would not be a ____________.

Because the square root is ______________ for _____________ numbers, all the _____________ real numbers are excluded from the ____________________ of the parent function. We need to make sure that all square roots have only ____________ numbers or ________ under them.

[1] We also only consider real-valued functions in this class. So, even though we know that $\sqrt{-1} = i$, for instance, we'll treat is as undefined in this section.

Example 1 Find the domain and range of $f(x) = -2\sqrt{x+4} + 6$.

Example 2 Find the domain and range of $g(x) = \sqrt{-6(x-2)} + 5$.

By applying ____________________ to the parent function, we get the __________________ of the square root function:

Recall from section 1.4 that n represents

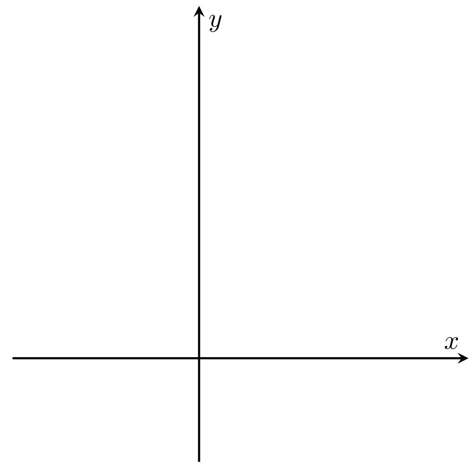
- a reflection across the y-axis if __________________
- a stretch from the y-axis by a factor of $\dfrac{1}{|n|}$ if ________________
- a compression toward the y-axis by a factor of $|n|$ if ____________

For our previous parent functions, their symmetry meant that all reflections could be represented with only A. This function has no symmetry, so n is needed as well.

A sketch of a square root function should include:

shape of curve	
x-intercept	
y-intercept	
endpoint	

Example 3 Sketch a graph of $f(x) = -2\sqrt{x+4} + 6$.

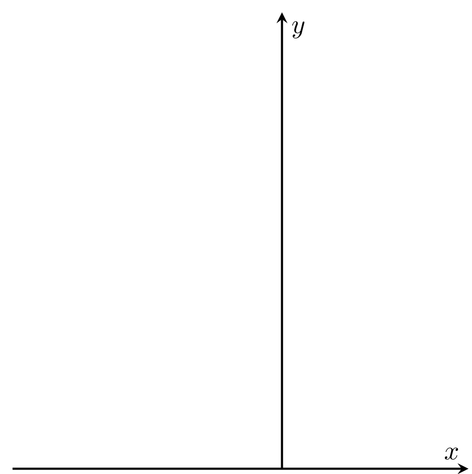


x-intercept:

y-intercept:

endpoint:

Example 4 Sketch a graph of $g(x) = \sqrt{-6(x-2)} + 5$.



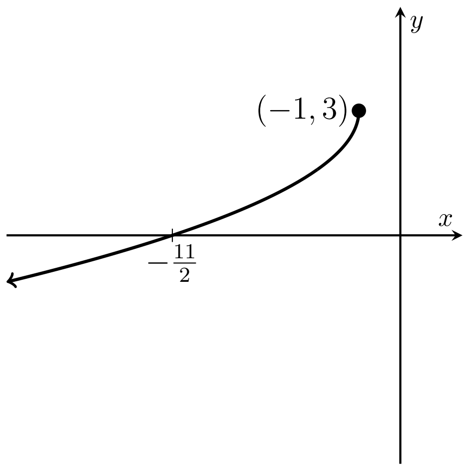
x-intercept:

y-intercept:

endpoint:

Example 5 List the transformations required to transform $f(x) = x^{1/2}$ to $g(x) = (-2x+5)^{1/2} - 3$.

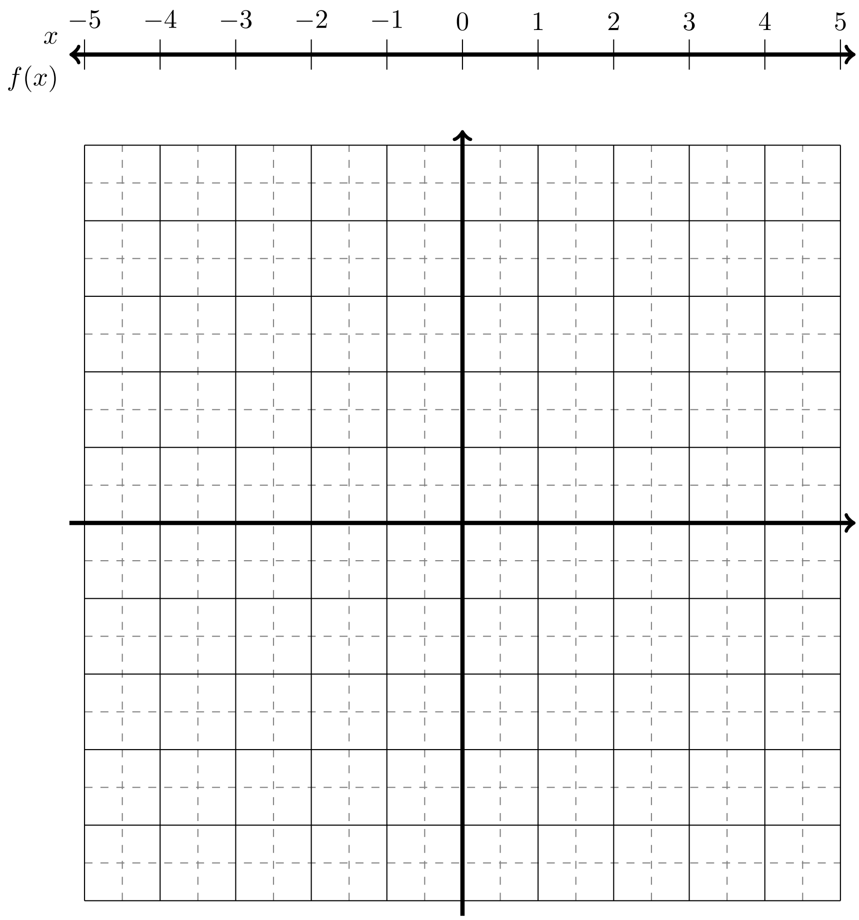
Example 6 Find the function f represented by the following graph.



Example 7 The parent function $f(x) = \sqrt{x}$ is compressed toward the x-axis by a factor of 5. What horizontal transformation results in the same function?

7.5 Cube Root Functions

parent function
domain
range
relation type
x-intercept
y-intercept
point of inflection[2]



Unlike the square root, the cube root can be evaluated for ____________ real numbers, which simplifies finding the __________ and _________ for cube root functions, which are both ______________________ if there is no domain restriction.

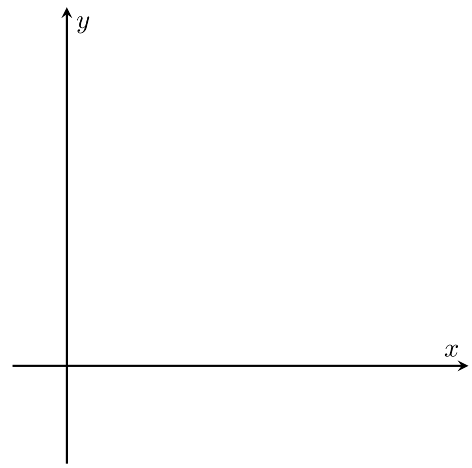
As the inverse of the ________ parent function, $y = x^3$, the curve of the ______________ function has the same shape, _____________ over the line $y = x$.

Using ____________________, we can write the general form for a cube root function

[2]This point does fit the definition of inflection we've used, because the curve changes from concave up to concave down here, but there are other ways to define inflection which would technically exclude this point. The distinction doesn't matter in this class, but does in Calculus. Alternatively, this could be called a *vertical tangent point*.

Example 1 Sketch a graph of $f(x) = -3\,(x-8)^{1/3}+6$.

point of inflection:

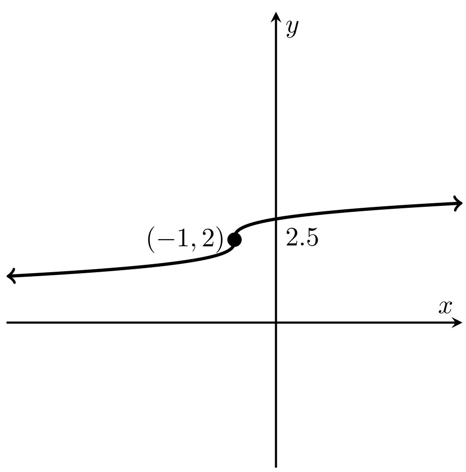


x-intercept:

y-intercept:

endpoints:

Example 2 Find the function g represented by the following graph.



7.6 Quadratics, Cubics and Roots as Inverses

Recall the following theorem:

Theorem

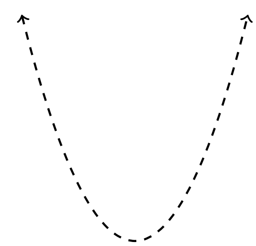
A function f has an ________________ f^{-1}
if and only if f is a ____________ function.

A cubic function of the form $f(x) = A(x-h)^3 + k$ is ____________, so it will always have an _________________. The _________ will be a ___________ function.

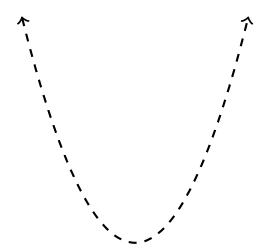
A quadratic function is more challenging because it is ______________, so does not have an _________________.

To get around this problem, we can restrict the ________ of the function.

The resulting _________ will be a _____________ function.



domain $= [h, \infty)$



domain $= (-\infty, h]$

Theorem

Suppose f is a __________ function,
and that $y = f(x)$ has a ________ at (h, k).

If the domain of f is _________ or ___________,
then f is ____________.

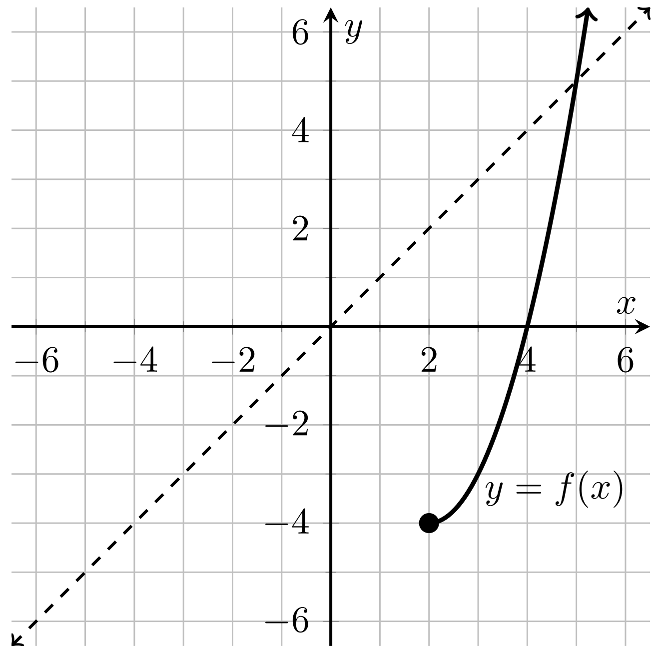
It is easiest to find the inverse of a quadratic functions in ________ form.

Example 1 Consider the function $f : [2, \infty) \to \mathbb{R}$, where $f(x) = (x-2)^2 - 4$.

a) Show that the inverse function f^{-1} exists.

b) Find the range of f, and hence, the domain of f^{-1}.

c) Find the rule for f^{-1}.



d) Use the graph of $y = f(x)$ shown to plot $y = f^{-1}(x)$ on the same plane.

Example 2 Find the inverse function of $g(x) = -2\sqrt{x-5}+3$, and state the domain and range for each of g and g^{-1}.

Example 3 Find the inverse function of $f(x) = [5(x+4)]^{1/3} - 9$.

Example 4 Find the inverse function of $g(x) = -\frac{3}{4}(2x-7)^3 + 5$.

Chapter 8

Exponential and Logarithmic Functions

8.1 Exponential Functions

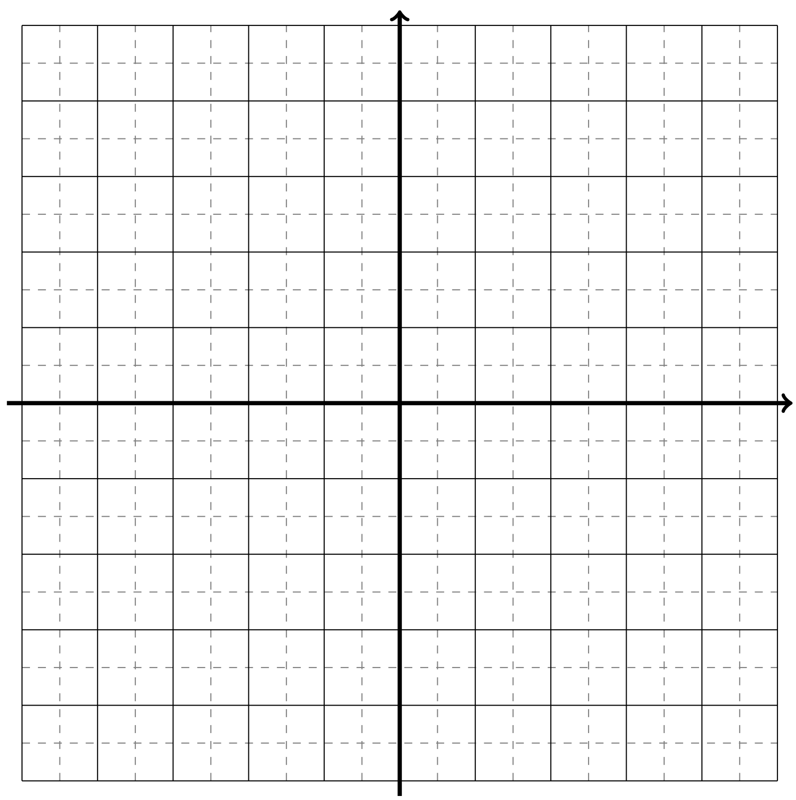
An __________________________ is a function of the form

where the ________, b, is a positive real number which is not 1. The simplest cases have $A = 1$ and $k = 0$, such as with the following two examples.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$											

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(x)$											

functions
domain
range
relation type
x-intercept
y-intercept
horizontal asymptote



For $b > 1$, including $b = 2$ above, the function shows _________________________, which means as the function increases, the rate of increase is also increasing proportionally.

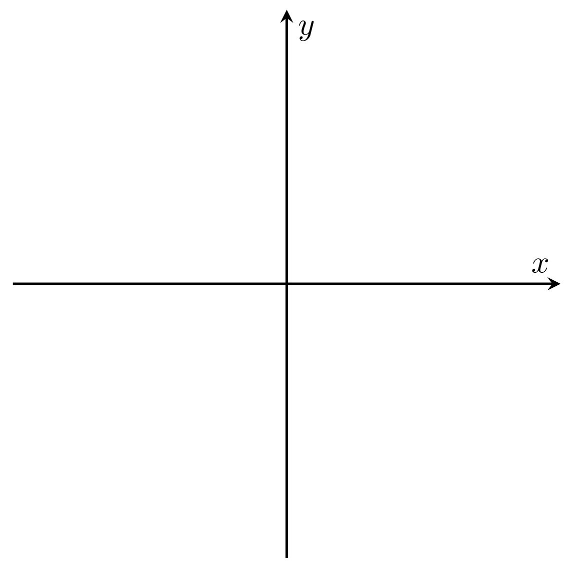
For $0 < b < 1$, including $b = \frac{1}{2}$ above, the function shows _________________________, which means as the function decreases, the rate of decrease is also decreasing proportionally.

A sketch of an exponential function should include:

shape of curve	
x-intercept	
y-intercept	
asymptote	
endpoints	

It is a good idea to show an additional point, such as $(1, f(1))$, to show the rate of growth or decay.

Example 1 Sketch a graph of $f(x) = \frac{1}{2}3^x - \frac{9}{2}$.



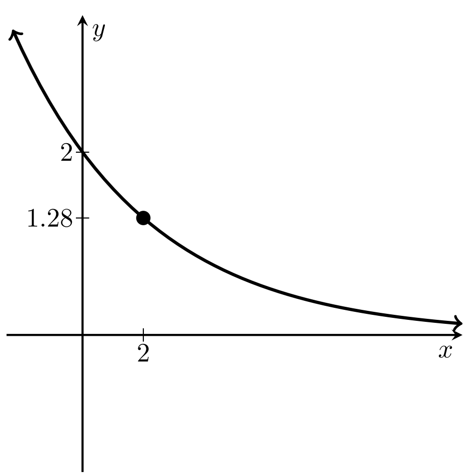
x-intercept:

y-intercept:

asymptote:

endpoints:

Example 2 Identify the function g represented in the graph below.

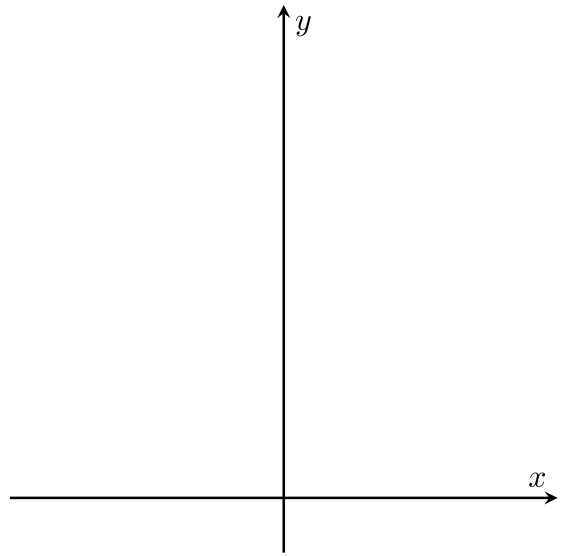


asymptote:

y-intercept:

point:

Example 3 Sketch a graph of $g(x) = 4^{\left(\frac{3}{2}x-1\right)} + 1$.



x-intercept:

y-intercept:

asymptote:

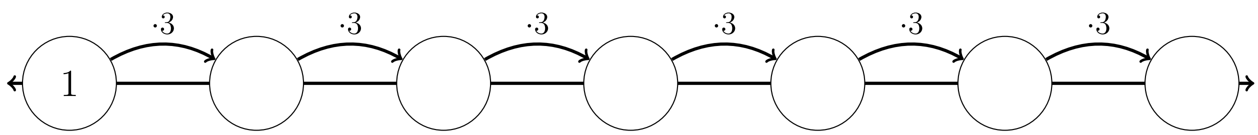
endpoints:

Example 4 Suppose f is an exponential function, whose graph $y = f(x)$ passes through the points $(2, 2)$ and $\left(5, \frac{1}{4}\right)$, and has an asymptote $y = 0$. Find the rule for $f(x)$.

8.2 Logarithms

Consider the equation $3^x = 243$, whose solution is the answer to the question

The diagram illustrates that the solution is



The mathematical operation which answers the question above is the ______________. This particular case is written

which is read as "the ______________ _________ 3 of 243." In general,

Example 1

$\log_5 125 =$ because

$\log_2 256 =$ because

$\log_4 \frac{1}{16} =$ because

$\log_7 \sqrt{7} =$ because

Note that if the base is omitted, it is assumed to be _____. This is sometimes known as a ____________ logarithm.

Example 2

$\log 10000 =$ because

$\log 0.001 =$ because

Example 3 Write the following equations in logarithmic form.

$a = 3^b$ $s = t^k$ $p = 10^r$

Example 4 Write the following equations in exponential form.

$u = \log_2 v$ $m = \log n$ $w = \log_y z$

Logarithm Rules

Recall that we reviewed the ____________________ in section 7.2. Some of those rules can be rewritten as equivalent ____________________.

Exponent Product Rule $$a^m \cdot a^n = a^{m+n}$$	**Logarithm Product Rule**
Exponent Quotient Rule $$\frac{a^m}{a^n} = a^{m-n}$$	**Logarithm Quotient Rule**
Exponent Power Rule $$(a^m)^n = a^{mn}$$	**Logarithm Power Rule**
Negative Exponent Rule $$a^{-n} = \frac{1}{a^n}$$	**Reciprocal Logarithm Rule**
Exponent Special Values $$a^0 = 1 \qquad a^1 = a$$	**Logarithm Special Values**

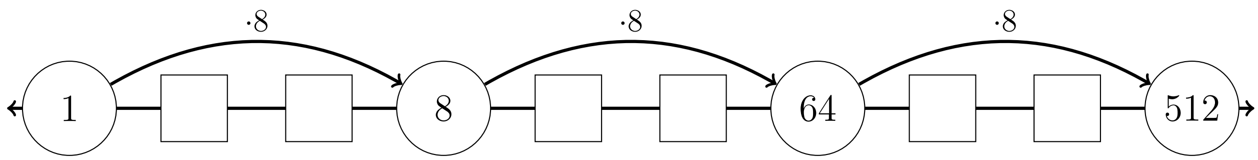
Example 5 Simplify the following without using a calculator.

$2\log_6 3 + \log_6 4$

$\log_5 8 - \log_5 1000$

The Change of Base Rule

Recall from section 7.2 that we used the following diagram to illustrate $8^{7/3} = 128$:



We can state this in logarithmic form as

When we originally calculated this, it was difficult to think of ______ as a power of ____. Instead, we expressed both numbers using _____ as the _________, which in logarithmic form are

Equivalently, we can write

This is an example of the following rule:

Theorem: Change of Base Rule

Example 6 Use the change of base rule to simplify the following.

$\log_{27} 81$

$\log_{25} \sqrt[3]{5}$

8.3 Logarithmic Functions

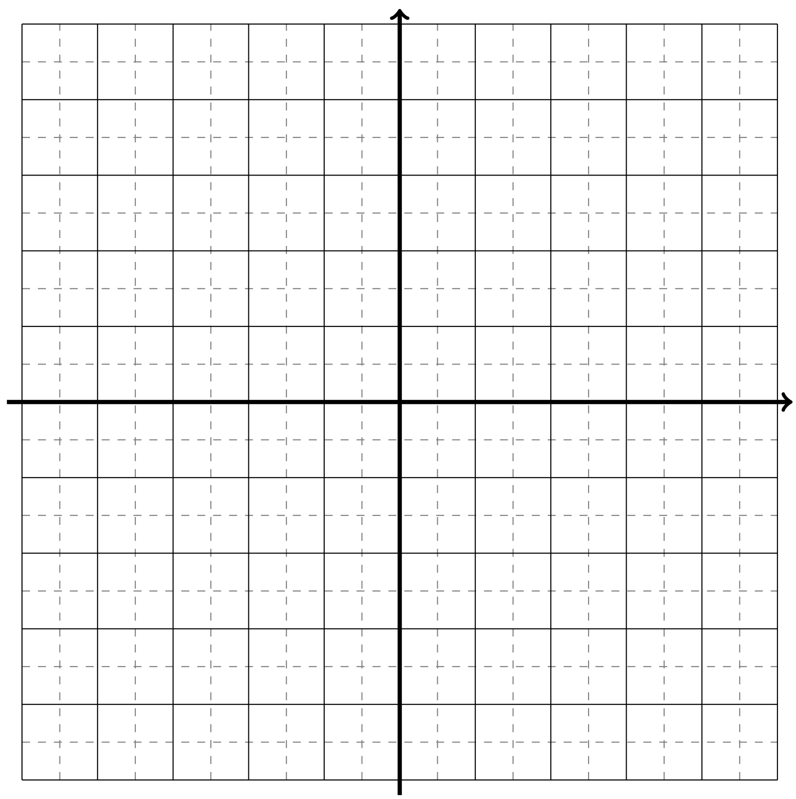
A ____________________________ is a function of the form

where the _________, b, is a positive real number which is not 1. The simplest cases have $n = 1$ and $h = 0$, such as with the following two examples.

functions	
domain	
range	
relation type	
x-intercept	
y-intercept	
vertical asymptote	

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$											

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(x)$											



Example 1 Express $f(x) = \log_5(x) + 2$ in the form stated above.

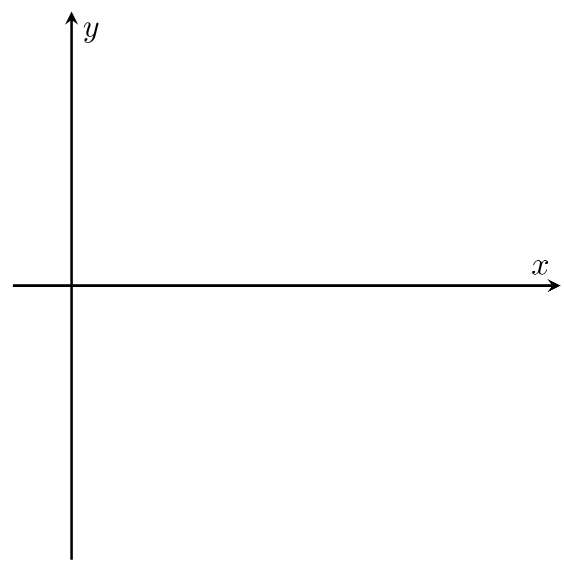
Example 2 Express $g(x) = \frac{1}{3}\log_2(x)$ in the form stated above.

A sketch of an logarithmic function should include:

shape of curve	
x-intercept	
y-intercept	
asymptote	

Example 3 Sketch a graph of $f(x) = \log_2 \left[\frac{1}{3}(x-4)\right]$.

x-intercept:

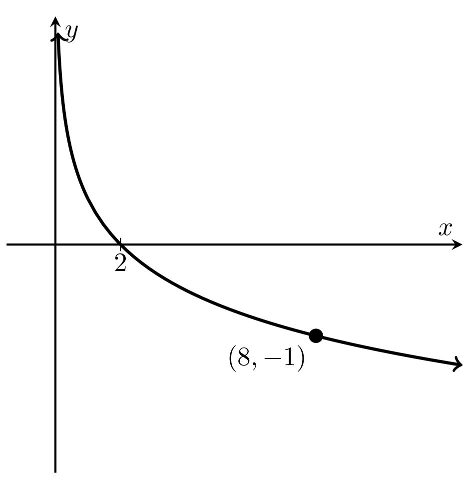


y-intercept:

asymptote:

other point:

Example 4 Identify the function g represented in the graph below.



asymptote:

x-intercept:

point:

Exponential and Logarithmic Functions as Inverses

The ___________ of an exponential function is a _______________ function with the same ________.

This means that the inverse of $f(x) = a^x$ is ______________________.

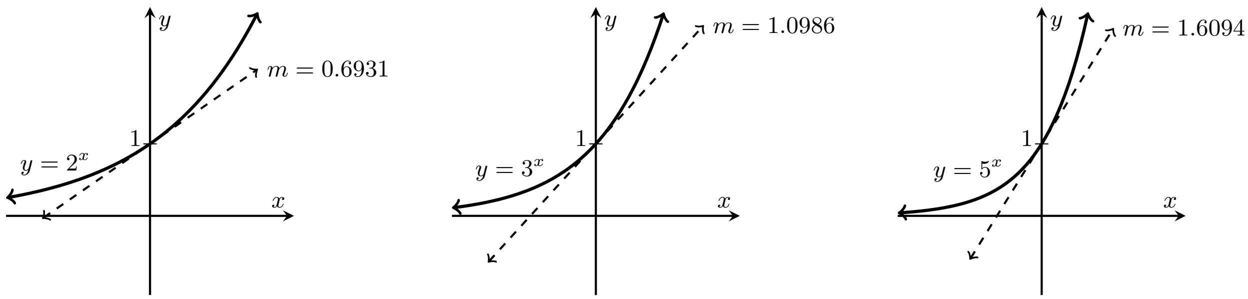
Example 5 Find the inverse function of $f(x) = 15 \cdot 3^x + 2$, and state the domain and range for each of f and f^{-1}.

Example 6 Find the inverse function of $g(x) = \log\left[6\left(x - 4\right)\right]$, and state the domain and range for each of g and g^{-1}.

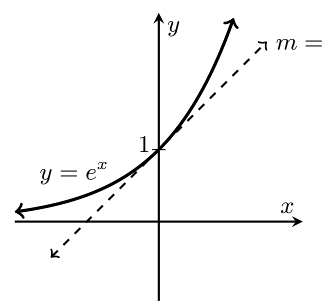
8.4 Natural Exponents and Logarithms

The Base e

Observe the following graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$.



You should recall that changing the ________ of the exponent does not change the ________________, which is $(0, 1)$ for each curve. However, changing the ________ does change how steep the curve is at this point. This is represented by the dashed line, which is the ___________ to the curve at the y-intercept.[1] Notice that the __________ of these tangents are decimal values, which each turn out to be irrational.



We might wonder if it's possible for the slope of this tangent to have an exact integer value, such as 1. As it happens, this occurs when the ________ is a particular _____________ constant, which we denote e, and has the value

$$e = 2.71828182845904523536\ldots$$

The relationship between a function and the slopes of its tangents is the basis for much of calculus, which makes the function $f(x) = e^x$ very important. e shows up in many other areas of math also, as well as being used in science, engineering, finance and many other applications.

For Algebra 2, we need to know of the existence of e and that it is closely related to exponents and logarithms. However, we don't need to worry if we don't yet understand why it is important or where it comes from.

When exponents or logarithms have e as their ________, they are called ___________. All exponential and logarithmic functions can be written as transformations of ___________ exponents and logarithms, so we can use these as ______________________.

The ______________________ is important enough that it gets its own notation:

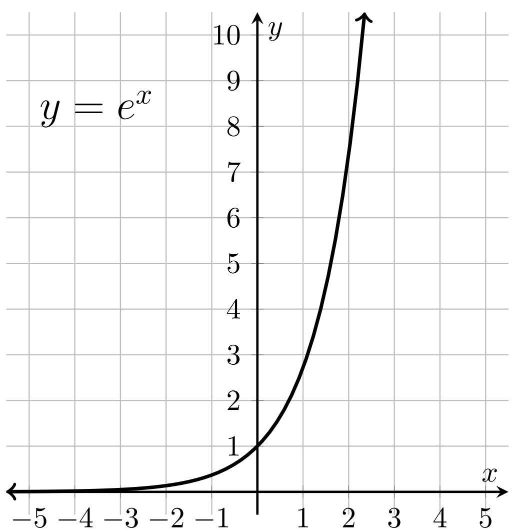
[1] Remember from Geometry that the tangent to a circle is a straight line which touches the circle at a single point? Graphs of functions also have tangents, which have a very similar meaning.

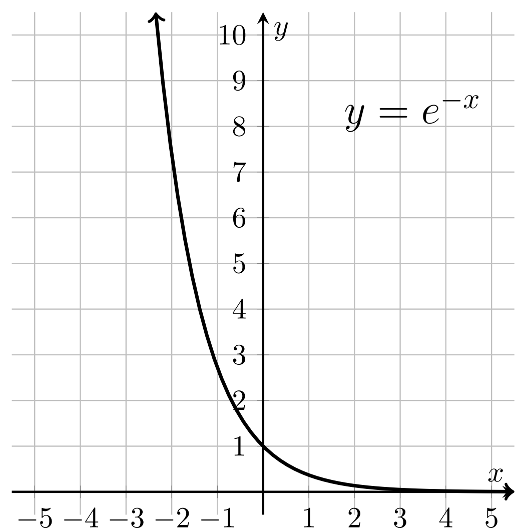
Natural Exponents

The parent function for natural exponents is _______________, which leads to the general form

Instead of changing the ________ to control the rate of exponential ___________ or __________, we can change the value of n. If n is ___________, the function exhibits exponential ___________. If n is _____________, the function exhibits exponential _________.

Example 1 Plot the points at $x = 0, 1, 2$ on each of the following graphs, and label them with exact coordinates.



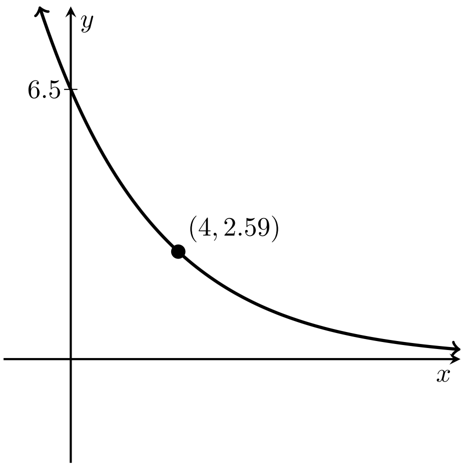


Since e^x and $\ln x$ are ____________, we can use the result $e^{\ln a} = a$ to change the base of an exponent to e:

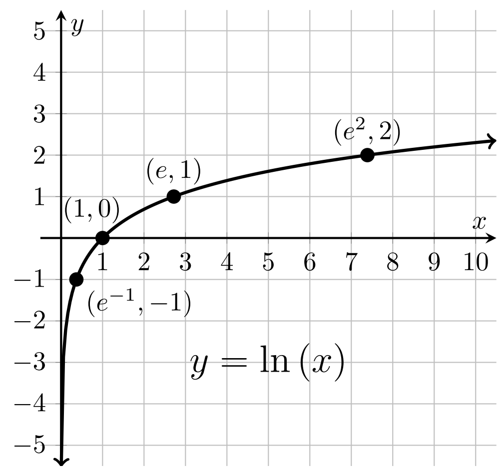
Example 2 Express $f(x) = 5 \cdot 4^x$ using e as the base.

Example 3 Express $g(x) = 3 \cdot \left(\frac{1}{8}\right)^x$ as a natural exponential function.

Example 4 Identify the function f represented in the graph below.



Natural Logarithms



The parent function for natural logarithms is ________________, which leads to the general form

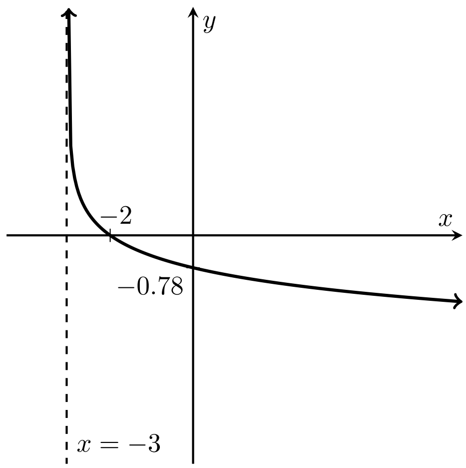
Instead of changing the ________ to control the ____________ and __________ of the logarithmic curve, we can change the value of A.

We already have the ____________________________ which we can use to change logarithms to their natural form:

Example 5 Express $f(x) = \log_4 3x$ using the natural logarithm.

Example 6 Express $g(x) = \log_{0.2} x$ using the natural logarithm.

Example 7 Identify the function g represented in the graph below.



Example 8 Find the inverse function of $f(x) = 20e^{-0.001x} + 5$. State the domain and range of each f and f^{-1}.

8.5 Exponential and Logarithmic Equations

Method 1: Equating the Base

The simplest method to solve equations involving ______________ or ______________ is often to write ______________ with the same ________. Then we can use the following theorem.

Theorem

Two exponential expressions with the same _______ are _______ iff (if and only if) they have the same ___________.

Example 1 Solve $81^{2x+1} = \sqrt{3}$.

Example 2 Solve $6^{5x+3} = 36^{4x+9}$.

This applies equally to ______________, as they are the ___________ of ______________. You'll need to check for extraneous solutions.

Example 3 Solve $\log(4x-2)-\log(x-5) = 1$.

Example 4 Solve $2\ln(x) = \ln(2x+3)$.

Method 2: Using Inverse Operations

Since exponents and logarithms are ____________ of each other, we can use them to solve equations involving the other. The solutions obtained when using this method are often ____________.

Example 5 Solve $\log_3(x+9) = 2$.

Example 6 Solve $3e^{x/4} + 4 = 10$ exactly.

Example 7 Solve $4^{2x-3} = 20$ to 2 decimal places.

Example 8 Solve $2\ln(x-1) + 5 = 1$ to 3 decimal places.

Method 3: Using a Substitution

Sometimes we can change an equation to a simplified form using a thoughtful ________________.

Example 9 Solve $3^{2x} - 6 \cdot 3^x - 27 = 0$.

8.6 Exponential Regression

Recall that ______________ is the process of fitting a modeling function to a set of data in order to approximate the relationship between variables.

____________________________ uses an ______________ function for the modelling function.

This means choosing values for ____ and ____ so that __________________ fits the data as well as possible.

Like linear and quadratic regression, performing ____________________________ involves calculating

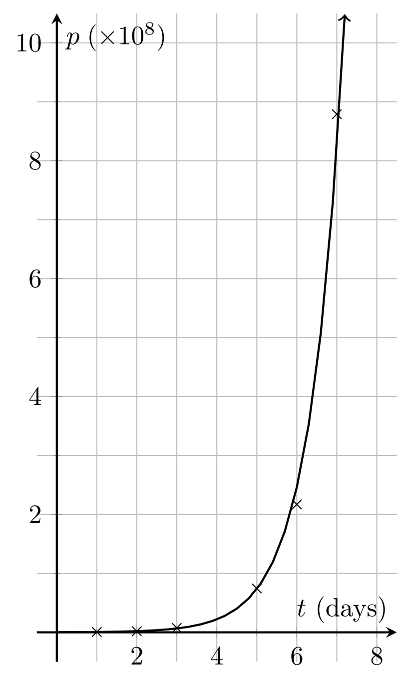
the the _________________________________, denoted by ______, which measures how well the regression curve fits the data.

If your device or software offers "log mode" for this type of regression, this generally provides a better fit. Some devices do this by default.[2]

Example 1 A research lab is investigating the population of a sample of bacteria. After leaving the sample for 24 hours at a time, the number of bacteria is estimated and recorded. Let t be the number of days after the beginning of the experiment.

t (days)	1	2	3	5	6	7
p	5.74×10^5	1.85×10^6	7.49×10^6	7.43×10^7	2.17×10^8	8.79×10^8

Use exponential regression to model bacteria population.



Example 2 Predict the population at the beginning of the experiment.

Example 3 The researchers weren't able to collect data on day 4. Estimate what the population would have been that day.

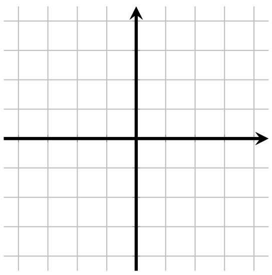
[2]How this works, and the reasons why performing exponential regression this way is preferable, are beyond the scope of Algebra 2.

Chapter 9

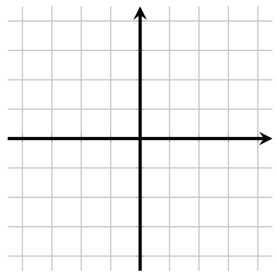
Further Functions

9.1 Identifying Functions

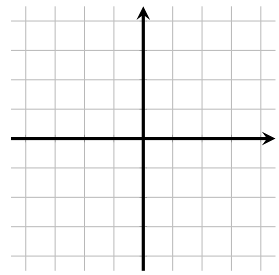
Review of Parent Functions



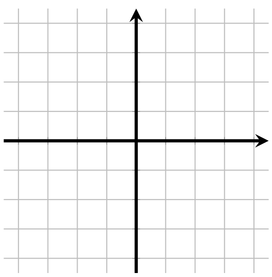
$f(x) = x$



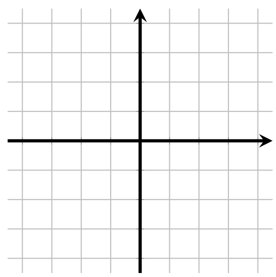
$f(x) = |x|$



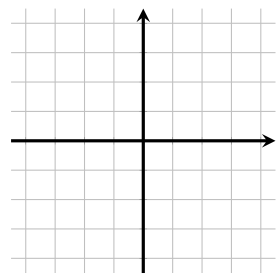
$f(x) = x^2$



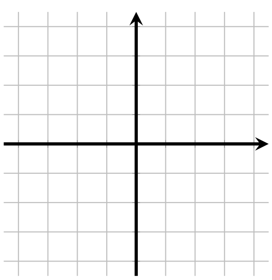
$f(x) = x^3$



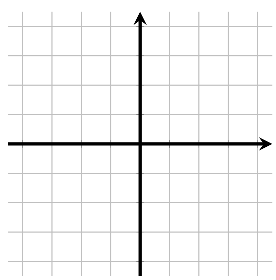
$f(x) = \frac{1}{x}$



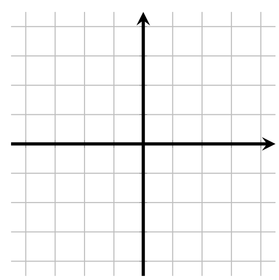
$f(x) = \frac{1}{x^2}$



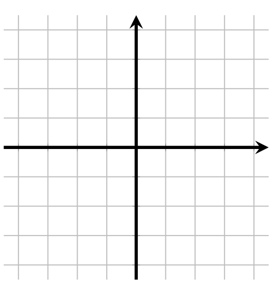
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



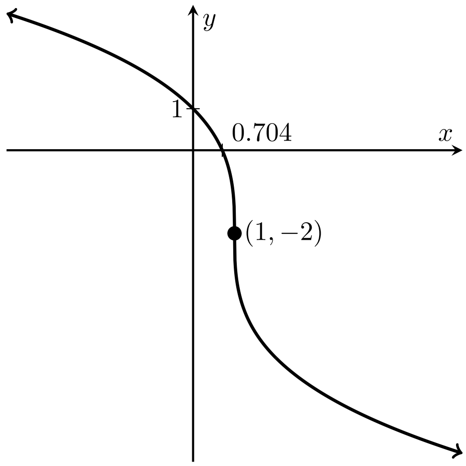
$f(x) = e^x$



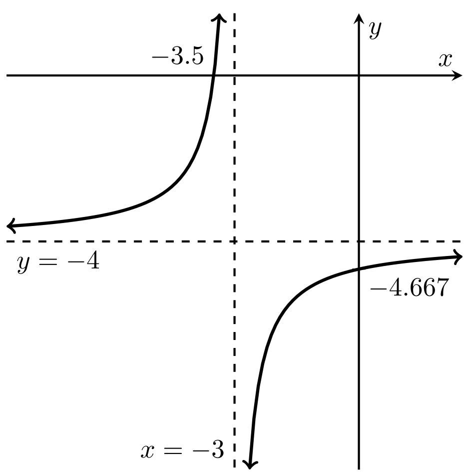
$f(x) = \ln x$

Recall that we can use these ________________________, together with ______________________, to construct functions. By identifying these in a ___________, we can identify the corresponding ______________.

Example 1 Identify the function f represented in the graph below.



Example 2 Identify the function g represented in the graph below.



9.2 Algebraic Combinations of Functions

By ______________ functions in a variety of ways, we can create ____________________. The simplest thing we can do is to _______, ____________ or ___________ functions.

- If $h = f + g$, then $h(x) = f(x) + g(x)$ for each value of x.
- If $h = f - g$, then $h(x) = f(x) - g(x)$ for each value of x.
- If $h = f \cdot g$, then $h(x) = f(x)g(x)$ for each value of x.

Note that for each of these cases, $h(x)$ is only ___________ if both $f(x)$ and $g(x)$ are ___________. This means that the ___________ of h is the ________________ of the ____________ of f and g.

We can also _________ functions.

- If $h = f/g$, then $h(x) = \dfrac{f(x)}{g(x)}$ for each value of x.

In this case, we need to remember that we can't divide by ________. So $h(x)$ is only ___________ if both $f(x)$ and $g(x)$ are ___________, and $g(x) \neq 0$.

Example 1 Complete the table.

x	-2	-1	0	1	2	3	4
$f(x)$	undef	2	6	0	1	3	-2
$g(x)$	3	0	2	4	undef	1	-2
$(f+g)(x)$							
$(f-g)(x)$							
$(f \cdot g)(x)$							
$(f/g)(x)$							

Example 2 State the domains of all of the functions in example 1.

Example 3 State the rule for $h = f + g$ if $f(x) = \ln(x+3)$ and $g(x) = \dfrac{1}{x-5}$. Find the domains of f, g and h.

In the previous example, the domain of the combined function could be identified from its rule as the implied domain.

In the following examples, we'll find that the domain of the combined function is different from the domain implied by its rule.

Example 4 Find and simplify the rule for $w = u \cdot v$ if $u(x) = \dfrac{1}{x+1}$ and $v(x) = x^3 + 3x^2 + 3x + 1$. Find the domains of u, v and w.

Example 5 Find and simplify the rule for $h = f/g$ if $f(x) = (x+3)e^{-x}$ and $g(x) = x^2 - 4x - 21$. Find the domains of f, g and h.

9.3 Function Composition

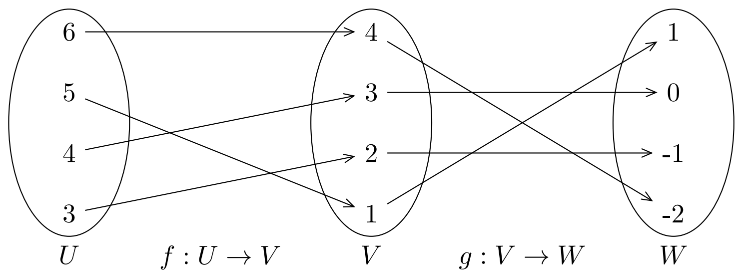
Another way to combine functions is ________________, which means using the __________ of one function as the ________ of another. The ________________ of f and g is denoted $f \circ g$, and the function is defined as

Note that the _________ matters, because ______________ f and g results in a different function.

Example 1

a) Complete the mapping diagram for $g \circ f$.

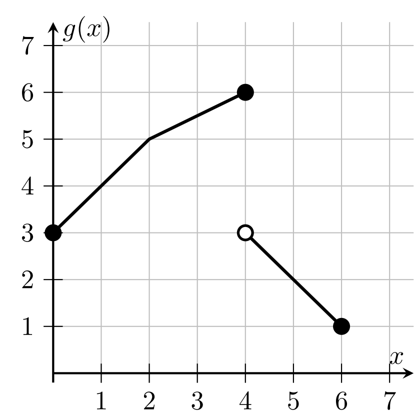
b) Are there any values for which $f \circ g$ is defined?



$g \circ f : U \to W$

Example 2 Use the function definitions to evaluate the compositions.

x	$f(x)$
0	4
1	3
2	0
3	1
4	5
5	6
6	2



$(f \circ g)(5)$

$(g \circ f)(3)$

$(f \circ g)(6)$ $(g \circ f)(2)$ $(g \circ g)(2)$

$(f \circ f)(0)$ $(g \circ g)(3)$ $(f \circ g)(3)$

Example 3 $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$. Find $g \circ f$ and $f \circ g$.

Example 4 $f : [-3, 6] \to \mathbb{R}$ where $f(x) = x^2$, and $g : (0, 11) \to \mathbb{R}$ where $g(x) = x - 7$. Find $f \circ g$, and find its domain and range.

Composition with the Inverse

With _______________, we can show that two functions are ____________, using the following theorem.

Theorem

$$f : A \to B \text{ and } f^{-1} : B \to A \text{ are \_\_\_\_\_\_\_\_\_ functions}$$
$$\text{iff } (f^{-1} \circ f)(x) = f^{-1}[f(x)] = x \text{ for every } x \in A$$
$$\text{and } (f \circ f^{-1})(x) = f\left[f^{-1}(x)\right] = x \text{ for every } x \in B$$

Example 5 Show that $f(x) = 5e^x - 8$ and $g(x) = \ln\left[\frac{1}{5}(x+8)\right]$ are inverses.

Example 6 Show that $f : [4, \infty) \to \mathbb{R}$ where $f(x) = x^2 - 8x + 21$ and $g(x) = \sqrt{x-5} + 4$ are inverses.

9.4 Piecewise Functions

We previously discussed piecewise functions in section 2.5, but only considered functions with _________ pieces. In general, any function can be a piece of a piecewise function. For this course, we'll include _____________ and _______________ pieces.

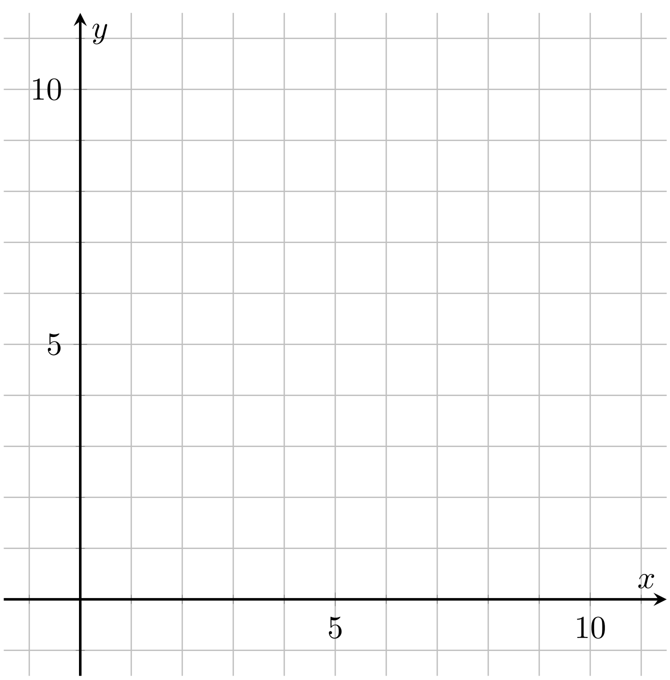
Example 1 Evaluate each of the following using the function f.

$$f(x) = \begin{cases} x^2 + 2 & 0 \le x < 3 \\ 16 \cdot 2^{-x} & 3 \le x < 6 \\ -x + 11 & 6 \le x < 10 \end{cases}$$

$f(1)$ $f(8)$ $f(5)$

$f(6)$ $f(3)$ $f(10)$

Example 2 For function f above, plot its graph and find its domain and range.



Domain:

Range:

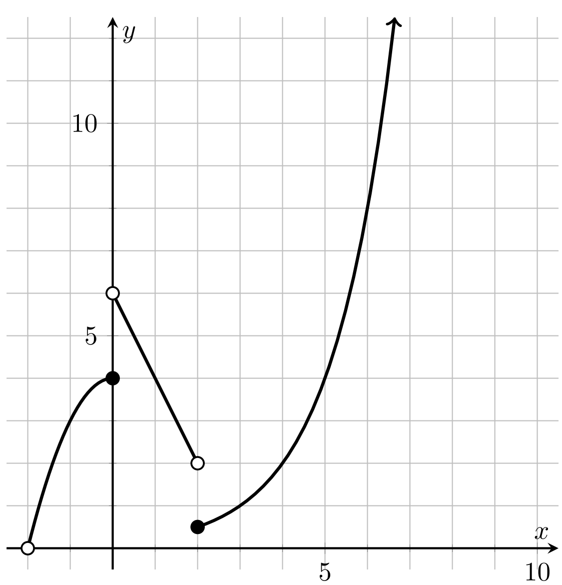
Example 3 Consider the function g defined as

$$g(x) = \begin{cases} x^2 - 8x + 12 & 1 < x \leq 5 \\ -3 & 5 < x < 8 \\ -x^2 + 20x - 99 & 8 \leq x \leq 13 \end{cases}$$

a) Find the zeros of g.

b) Find the intervals g is increasing, decreasing, or constant.

Example 4 Find the function h represented in the graph below.



Chapter 10

Matrices

10.1 Matrix Operations

A __________ (plural ____________) consists of numbers arranged into ________ and ___________ in a rectangle. It is typical to assign them ________________ variables, and to surround them with ____________.[1]

For example,

$$A = \begin{bmatrix} 3 & 7 & -2 \\ 9 & -4 & 1 \end{bmatrix}$$

The ______________ of a matrix denote the number of ________, m, by the number of ___________, n, which we write as ___________, and read as _____________.

For example, the ______________ of A above are _________, or we say A is a _________________.

The individual _____________ of a matrix are denoted by _______, where a is the lower case letter corresponding to the matrix variable, i indicates which _______, and j indicates which __________.

Example 1 Write the following using A above.

$a_{1,2}$ $a_{2,1}$ $a_{1,3}$

A matrix with the same number of ________ and ___________, or an __________________, is called a __________________.

An ___________________ is a square matrix with ________ along its ___________ (top-left to bottom-right), and _________ everywhere else. If the ___________________ is $n \times n$, it is denoted I_n.

Example 2 Write down I_3.

Example 3 If $B = I_7$, find $b_{4,2}$ and $b_{5,5}$.

[1] Some mathematicians prefer to use parentheses.

Adding and Subtracting Matrices

Matrices can be added or subtracted by adding or subtracting individual ____________ in ____________________________. This is only possible if the matrices have the same ______________, and the resulting matrix will also have the same ______________.

Example 4 If $C = \begin{bmatrix} 3 & 6 \\ -5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -7 & 8 \\ 2 & -4 \end{bmatrix}$, find $C + D$ and $C - D$.

Multiplying a Matrix and a Scalar

To distinguish them from matrices, individual numbers are called __________.

A _________ cannot be added to or subtracted from a matrix, but it can be ____________. To do so, we multiply each ___________ in the matrix by the scalar. The result is a _________ with the same ______________ as the original matrix.

Example 5 Using $A = \begin{bmatrix} 3 & 7 & -2 \\ 9 & -4 & 1 \end{bmatrix}$, find $-5A$.

Example 6 Find $3D - 4C$, using C and D above.

10.2 Solving Linear Systems with Matrices

We can take a system of linear equations at write them as a single matrix equation:

$$\begin{cases} a_{1,1}x & + & a_{1,2}y & + & a_{1,3}z & = & b_1 \\ a_{2,1}x & + & a_{2,2}y & + & a_{2,3}z & = & b_2 \\ a_{3,1}x & + & a_{3,2}y & + & a_{3,3}z & = & b_3 \end{cases} \qquad \longleftrightarrow \qquad AX = B$$

$$\text{where} \qquad A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then we can solve the matrix equation. The techniques used are beyond the scope of this course, and tedious to perform by hand anyway, but are simple for a calculator.

Reduced Row Echelon Form

Step 1: Write matrices A and B together, which is called an ______________ matrix.

$$\left[A \;\middle|\; B\right] = \left[\begin{array}{ccc|c} a_{1,1} & a_{1,2} & a_{1,3} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & b_3 \end{array}\right]$$

Step 2: Apply the operation _______ to the matrix using a calculator. This applies a series of operations which are equivalent to solving the system using the elimination method.

Step 3: Interpret the solution from the resulting matrix.

Example 1 Solve

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 11 \\ -x + 3y + 4z = 8 \end{cases}$$

Notice that A has been replaced with the ___________________. This will always happen if there is a ___________________ to the system. If not, then the matrix takes a different form.

Example 2 Solve

$$\begin{cases} 5x - 3y + z = -5 \\ 2x + y + 3z = 9 \\ 7x - 2y + 4z = 12 \end{cases}$$

Example 3 Solve

$$\begin{cases} 5x - 3y + z = -5 \\ 2x + y + 3z = 9 \\ 7x - 2y + 4z = 4 \end{cases}$$

Determinants

An important property of a ___________________ is its ________________. It is denoted by _________________ replacing the brackets around the matrix. The ________________ of a matrix A can be written _______ or ___________.

Determinant

The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The determinant can be found for larger $n \times n$ matrices, but becomes much more complicated. It is much easier to find using a calculator.

Example 4 Find the following determinants.

$$\begin{vmatrix} -3 & 2 \\ 4 & -1 \end{vmatrix} \qquad \begin{vmatrix} -1 & -4 \\ 3 & 2 \end{vmatrix}$$

The following result is particularly useful for linear systems.

Theorem

A linear system, written in the matrix form $AX = B$,

has a ________________ iff

Example 5 Confirm the nature of the solutions for the systems in the earlier examples.

Chapter 11

Sequences and Series

11.1 Introduction to Sequences and Series

Sequences

A ______________ is a collection of mathematical objects (in this class, numbers) in a specific _________. Unlike in _______, the numbers in a ______________ may be _____________.

Example 1 The sequence of all positive odd integers less than 20, in descending order, is

The individual entries in a sequence are known as __________. Each ________ can be identified using a lower case letter (we'll typically use _____) with a ______________ indicating its position in the sequence.

Example 2 Find each of the following for the sequence above.

a_1 a_3 a_6 a_{10}

If a sequence ends after a certain number of terms, it is _________. Otherwise, it is ___________.

While any numbers can be placed in an order to form a sequence, we're particularly interested in sequences which can be formed using a _______.

Explicit Rules

An ________________ calculates the value of each term using its position in the sequence.

Example 3 Calculate the first 6 terms of the sequence $a_n = n^2 + 1$.

n	calculation	a_n
1		
2		
3		
4		
5		
6		

Recursive Rules

The word ____________ refers to definitions or processes which refer to themselves in some way. A __________________ calculates the value of each term using the values of the previous term, or possibly multiple previous terms.

If we think of a_n as the __________ term, then a_{n-1} is the ___________ term, and a_{n+1} is the _______ term.

These rules require at least one ______________, a term that isn't defined ______________.

Example 4 Calculate the first 6 terms of the sequence $a_n = 2a_{n-1} - 3$, with $a_1 = 5$.

n	calculation	a_n
1		
2		
3		
4		
5		
6		

Example 5 List the first 10 terms of the Fibonacci sequence, defined as $f_n = f_{n-2} + f_{n-1}$, with $f_1 = f_2 = 1$.

Types of Sequences

An ________________________ has a constant ______________ between consecutive terms:

A ________________________ has a constant _______ between consecutive terms:

Example 6 Determine whether the following sequences are arithmetic, geometric or neither.

$$1, 5, 9, 13, 17, 21, \ldots$$

$$12, 6, 3, 1.5, 0.75, 0.375, \ldots$$

$$1, 2, 6, 24, 120, 720, \ldots$$

$$8, 8, 8, 8, 8, 8, \ldots$$

Sums and Sigma Notation

Recall that the _______ of a collection of numbers is the result obtained by __________ them.

Example 7 Find the sum of 2, 4, 6, 8, 10 and 12.

We can write this sum more concisely using the upper case Greek letter _________, Σ.

- Below Σ, we have the ______________________, k, and its _______________, 1.
- Above Σ, we have the ______________ of the indexing variable, 6.
- After Σ, we have the quantity to be summed, which is __________ the indexing variable in this case.

Example 8 Evaluate $\sum_{k=1}^{5} k^2$.

Example 9 Write $5 + 10 + 15 + 20 + \cdots + 100$ using sigma notation.

Series

A __________ is the sum of the first n terms of a sequence[1], which can be written as

Example 10 For $a_n = 3n + 5$, find S_8.

n	calculation	a_n	S_n
1			
2			
3			
4			
5			
6			
7			
8			

Example 11 For $a_n = 4a_{n-1} - 7$ with $a_1 = 3$, find S_5.

n	calculation	a_n	S_n
1			
2			
3			
4			
5			

[1]Mathematicians usually call this a *partial sum*, and reserve the word *series* for an infinite sum.

11.2 Arithmetic Sequences and Series

Recall that an ____________________ has a constant _______________ between consecutive terms:

Theorem

The recursive rule for an arithmetic sequence with difference d is

Example 1 Find the recursive rule for the sequence $5, 2, -1, -4, -7, \ldots$

Example 2 An arithmetic sequence begins with -2 and 4. State its recursive rule and find the first 8 terms of the sequence.

We can use the recursive rule repeatedly to find expressions for the terms following a_1.

a_2 a_3 a_4 a_5

Theorem

The explicit rule for an arithmetic sequence with difference d and first term a_1 is

The related function $f(n) = a_n$ is ________.

Example 3 Find the 50th term of the sequence $1, 5, 9, 13, 17, \ldots$

Example 4 In the sequence $a_n = a_{n-1} - 9$, $a_1 = 500$, which term is equal to 221?

Theorem

The finite series of an arithmetic sequence given by a_n is

Example 5 For $a_n = a_{n-1} - 4$, $a_1 = 88$, find the sum of the first 40 terms.

Example 6 Find the sum of the odd numbers between 0 and 200.

11.3 Geometric Sequences and Series

Recall that a __________________________ has a constant ________ between consecutive terms:

Theorem

The recursive rule for a geometric sequence with ratio r is

Example 1 Find the recursive rule for the sequence $\frac{1}{18}, \frac{1}{3}, 2, 12, 72, \ldots$

Example 2 An geometric sequence begins with -2 and 4. State its recursive rule and find the first 8 terms of the sequence.

We can use the recursive rule repeatedly to find expressions for the terms following a_1.

a_2 a_3 a_4 a_5

Theorem

The explicit rule for a geometric sequence with ratio r and first term a_1 is

The related function $f(n) = a_n$ is ______________.

Example 3 Find the 12th term of the sequence $640, 320, 160, 80, \ldots$

Example 4 Which term of the sequence $a_n = 5a_{n-1}$, $a_1 = 3$ is the first to be greater than 1 billion?

Theorem

The finite series of a geometric sequence given by a_n is

Example 5 For $a_n = \frac{1}{2}a_{n-1}$, $a_1 = 100$, find the sum of the first 8 terms.

Example 6 If the sum of the first 4 terms of $a_n = 3a_{n-1}$ is 480, what are those 4 terms?

Chapter 12

Data and Statistics

12.1 Statistical Concepts

In the field of statistics, a ____________ is a characteristic of a person or thing, which can have different values for each person or thing. A recorded value of a variable is called a __________, the plural of which is ________. The two main types of variables are

- _____________________________, whose data are numerical values for which it makes sense to use with arithmetic operations, and
- _____________________________, whose data place the people or things into groups or categories.

In this class, we'll mostly focus on quantitative variables and data.

Example 1 Decide if the following are quantitative or categorical.

- The salary of a software engineer. ________________
- The fur color of a pet cat. ________________
- The zip code of a customer. ________________
- The weight of a football player. ________________
- The number of students in an Algebra 2 class. ________________

In this section, we'll focus on ____________________, which is data for a single variable.

A ____________ is a single measure which summarizes a characteristic of a collection of data.

Measures of Central Tendency

A ____ is a statistic which uses a single number to represent an entire set of data.

- The _________ is the sum of the data values divided by their number:
- The ___________ is the value in the __________ when the data are ordered, or the _________ of the middle two values.
- The _________ is the ____________________ value.

Example 2 Find the mean, median and mode of 2, 3, 3, 3, 4, 7, 7 and 11.

Measures of Spread

A ____________________________ is a statistic which indicates how far the data _____________ from the ___________.

- The _____________ measures spread using the differences of each value from the mean, and is calculated with the formula:
- The _________________________ is the square root of the _____________, and is used more often as it shares the same _________ as the data:
- The __________ is the difference between the smallest and largest values.
- The __________________________, or _______, is the difference between Q_1 and Q_3, which are the medians of the lower and upper halves of the data respectively.

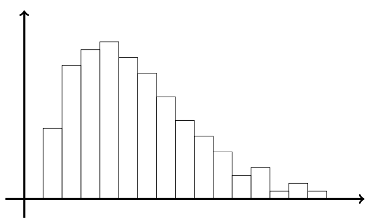
Example 3 Find the standard deviation of the values in the previous example.

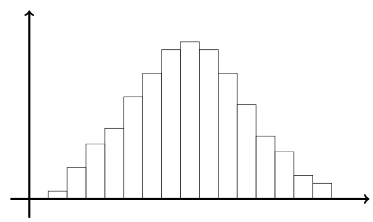
x	$x-\bar{x}$	$(x-\bar{x})^2$

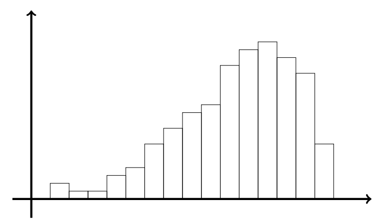
Skewed Distributions

Examining a _______________ representing a set of univariate data can reveal characteristics of the data.

If the bulk of the data is situated toward one end of its range, the data is said to be ____________. The direction of the _______________ is the same as the direction of the distribution's ______.



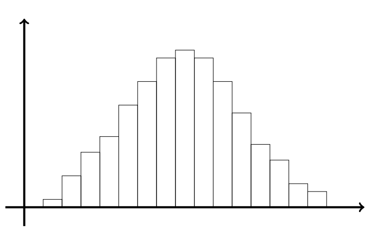


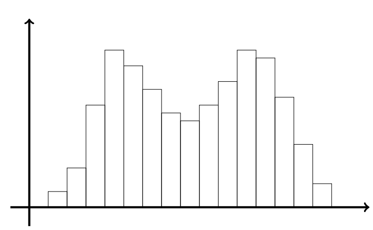


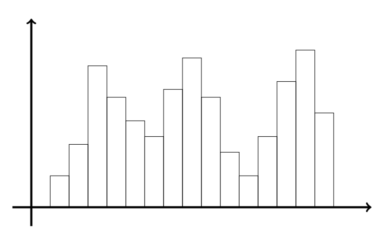
The __________ is affected by skewed values more than other measures of central tendency, so the relationship between __________ and ____________ can indicate the direction of any skewness.

Unimodal and Multimodal Distributions

Data distributions can also be characterized by the number of __________. It is typical to use the suffix _________ to refer to these, even if the peaks do not have the same height, and therefore do not strictly meet the definition of the _________.







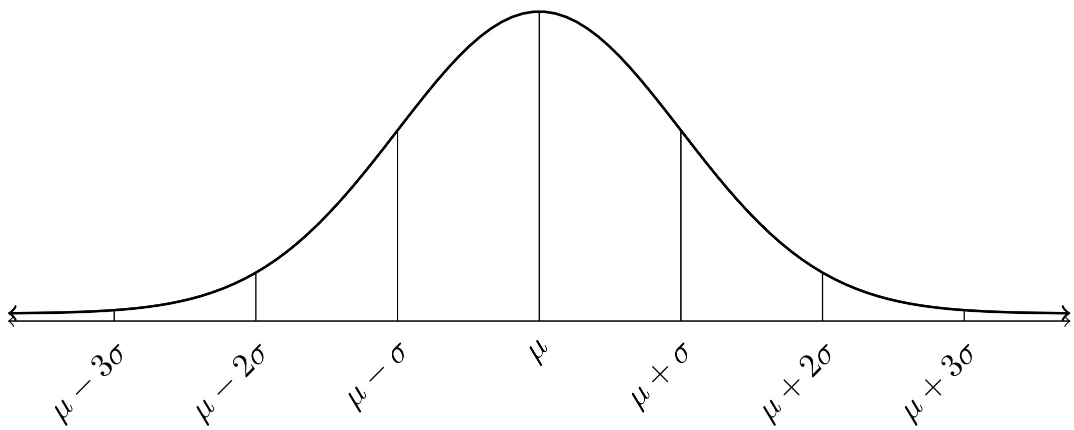
Distributions with more than one peak can also be called _______________.

12.2 Normal Distributions

A ________________________ is a type of probability distribution. Each normal distribution is defined by two _______________:

- The _________, represented by μ (lower case Greek letter mu).
- The _______________________, represented by σ (lower case Greek letter sigma).

The normal distribution can be graphed using a __________________, which is sometimes called a ______-shaped curve. The area under the curve can be interpreted as probabilities in the related normal distribution.



- The distribution is ____________, as it has one mode at the _________.
- The distribution is ______________ about the _________. _______ of the area is less than the _________, and _______ is greater than the _________.
- The **68-95-99.7 rule** states that
  - about _______ of the area is within _______ standard deviation of the mean,
  - about _______ of the area is within _______ standard deviations of the mean, and
  - about _________ of the area is within _________ standard deviations of the mean.

If a univariate data set is ____________ and ______________, then it may be appropriate to use a normal distribution to _________ the data. We can fit the distribution to the data by choosing parameters

Note the different symbols for mean and standard deviation. While we often choose them to have the same values, they have different meanings. $\bar{x}$ and s are the ____________ calculated from the _______, while μ and σ are the ________________ of the distribution.

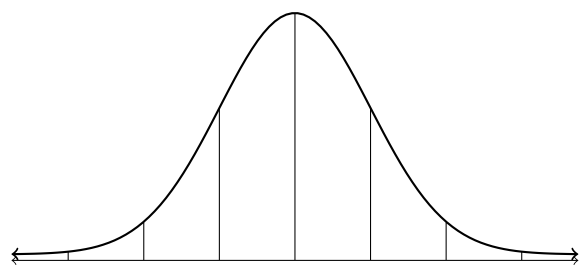
If X is a random variable, then we can use the notation

to represent:

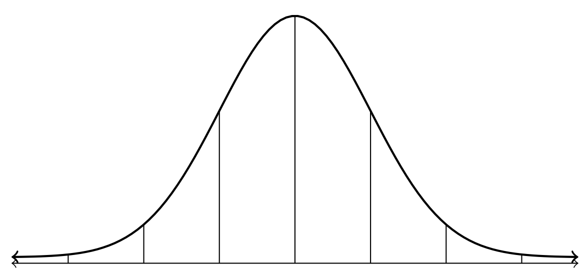
- The _______________ of individuals whose values which fall between a and b.
- The _______________ that an individual chosen at random has a value between a and b.

Example 1 The heights of a group of students are normally distributed with a mean of 5 ft 9 in and a standard deviation of 1.5 in.

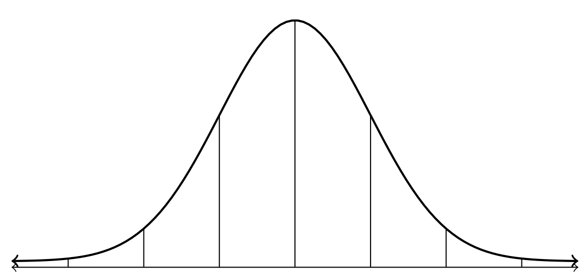
a) Find the proportion of students whose heights are between 5 ft 7.5 in and 6 ft.



b) Find the probability that a randomly chosen student is taller than 5 ft 6 in.



Example 2 In a normally distributed data set, 84% of the data values are less than 29, and 2.5% of the data values are less than 17. What are the mean and standard deviation?



12.3 Bivariate Data

When data is collected for two variables from the same set of subjects, it is called ____________________. In these cases, our interest is in knowing if there is an ________________ between the variables, which means that changes in one variable tend to occur with changes in the other.

Review of Regression

A key tool we have for examining bivariate data is ________________, as we've studied previously.

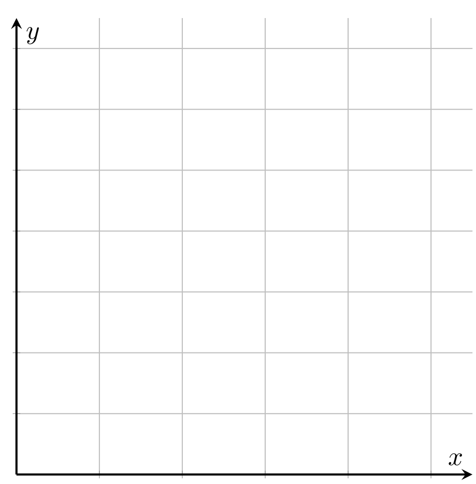
While we've used __________, ______________ and _________________ regression, and we'll continue to restrict ourselves to those three for this class, regression is possible using any type of function for which an association could exist.

Recall:

- The aim of ________________ is to find a _____________ which ___________ an ________________ between variables.
- The ______________________________________, denoted by _______, is a number between 0 and 1 indicating how well the __________ fits the data, with _____________ indicating a perfect fit.
- The ______________________________, denoted by _____, is a number between -1 and 1 which indicates the ______________ and _____________ of the linear association between the two variables. For linear regression, ______________.

Example 1 Find a function to model the data below.

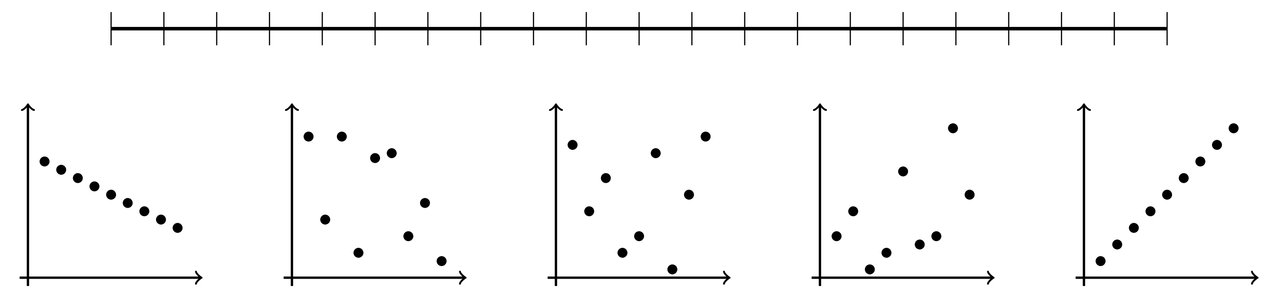
x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	5.0	4.3	3.9	3.7	4.1	5.0	6.3



Correlation and Causation

______________ measures a linear relationship between variables by indicating how one variable changes as the other variable increases.

If increases in one variable sees proportionally similar ______________ in the other, there is a ________________________________ between the variables, and r is close to _____. If increases in one variable sees proportionally similar ______________ in the other, there is a ________________________________ between the variables, and r is close to _______. In both cases, there is a ______________________________ between the variables.



Suppose that there are two variables, X and Y, which have a ________________________________. As stated above, this means that as X increases, Y also increases at a proportionally similar rate. This does not mean, however, that an increase in X __________ an increase in Y. There are actually three possibilities:

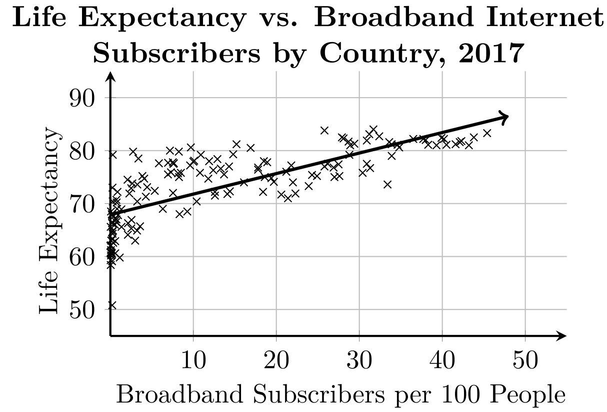
- Changes in X do indeed _________ changes in Y.
- The causation is ____________, and changes in Y _________ changes in X.
- Changes in X and Y are both __________ by changes in a _________________.

Not understanding this (or deliberately ignoring this) leads many people to make ________________ not supported by the data. As you hear or read statistical conclusions made by others, or are trying to draw your own conclusions, it is vital to remember this principle:

Correlation vs. Causation

Example 2 This graph and the correlation coefficient $r = 0.7485$ show that there is a fairly strong positive correlation between the number of broadband internet subscriptions in a country and the life expectancy in that country.

Is it reasonable to say that if a country wants to raise life expectancy, they should improve their internet infrastructure?



Sources:
https://data.worldbank.org/indicator/IT.NET.BBND.P2
http://gapm.io/ilex

Discrete and Continuous Models

A quantitative variable which can take only distinct, countably-many values is called ____________. These values generally arise from a ____________ process.

A quantitative variable which can take any value within an interval is called ______________. These values generally arise from a ______________ process.

Distinguishing between the two is important for deciding how to create graphs modeling the variable.

Example 3 A local car dealer promises to sponsor the high school softball team \$500, plus \$150 for each run they score in the next game, up to a total sponsorship of \$2000. Create a graph relating sponsorship money to runs scored.

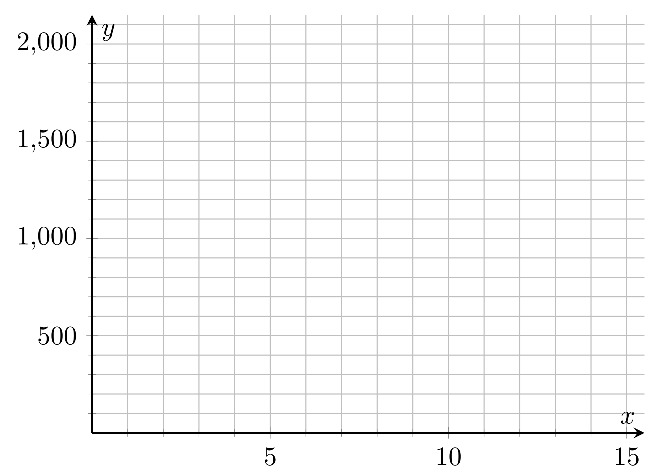
Independent Variable:

Dependent Variable:

Discrete/Continuous:

Domain:

Function:



12.4 Collecting and Presenting Data

The aim of ____________ is to understand __________ about the world through the collection and interpretation of _______. Every day, people form __________ and make _____________ based on the data that have been presented to them.

Unfortunately, data can be ____________ in ways that make them _____________, or can be ______________ in ways that are ______________. While some people will ______________ data in these ways deliberately, it is very easy to _______________ misuse data. Knowing how data can be misinterpreted helps us to avoid being _____________ by claims made by others, and to better _______________ the data we collect ourselves.

Populations and Samples

If we're interested in data regarding a particular class of people or things, the ______________ is the entire set of people or things in that class.

Example 1 A medical researcher is collecting data about the weights of 15 year olds in Oklahoma. What is the population?

If data are collected from every individual in the population, the process is called a ___________. This is ideal, as we know that the data truly represents the entire population. However, doing so is often impractical.

Instead, data are typically collected from a ___________, which is a subset of the population which is intended to represent the entire population. The sample should contain a ________ number of individuals to minimize the effect of random variation.

There are many different methods to select the sample, with varying quality. Here are a few common sampling methods:

- A ______________________________ selects the members of the sample from the entire population at random. This is usually best practice if possible. This can be as simple as drawing names from a hat, or can be done by assigning numbers to each individual and using a random number generator.
- A _______________________ places individuals into groups, then randomly selects members from every group. This ensures that every group is represented in the sample.
- A _______________________ places individuals into groups, then selects every member from randomly selected groups. This is often easier to administer, while still containing some randomness in the sample.

- A ______________________________ selects individuals who are willing to participate in a survey. Sometimes this is the only way to collect data, for legal or ethical reasons, but may introduce _______________.

- A _______________________ selects the individuals who are easiest to collect data from. This almost certainly introduces _______________. While this is a popular method because it is easy, informed statisticians should not use it.

Any factor that affects the data in a way such that they do not represent the true state of the population is called a _______. If the source of the _______ is the way the sample was selected, it is called _______________. Other _________ include __________________, which is where the presence of an ____________ affects the behavior or response of individuals in the sample.

Example 2 A business manager at a large company is concerned that many of her employees are spending a lot of time using social media when they should be working. She asks her assistant manager to conduct some research. He asks the first five people into the office the next day how much time they've wasted on social media. He reports to his boss that there is no social media problem at the company.

Are there any issues regarding the data collection in this scenario?

Recognizing Distorted Data Displays

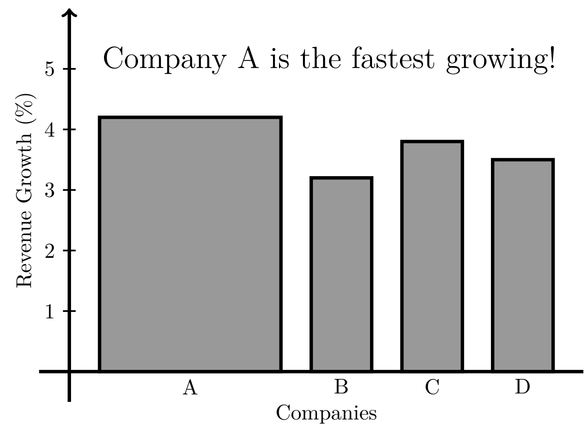
Presenting data in a _________ is a useful way to communicate and emphasize aspects of the data that are important to the author of the display. Unfortunately, it is possible to present data in ways that, while not false, are _____________.

An important rule to remember when presenting data is the __________________. This says that if a quantity is represented by a two-dimensional region in a graph, the ________ of the region should be _______________ to the quantity.

Example 3

This chart violates the ___________________ because the bars do not have the same _________. Even though Company A does have the highest growth, the difference in growth _____________ to be much greater because the bar's ________ is much greater.

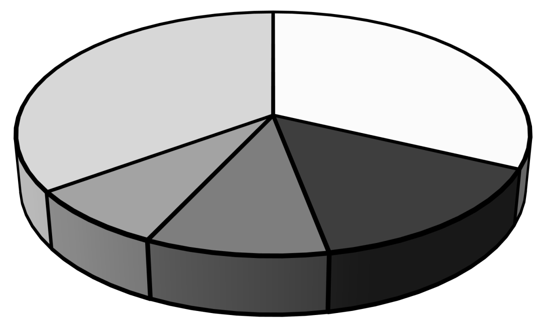
In general, the bars in a bar chart should all have the same _________.



Example 4

This chart violates the ___________________, because the _______________ on the pie chart causes some of the sectors to have additional _________________ along the edge.

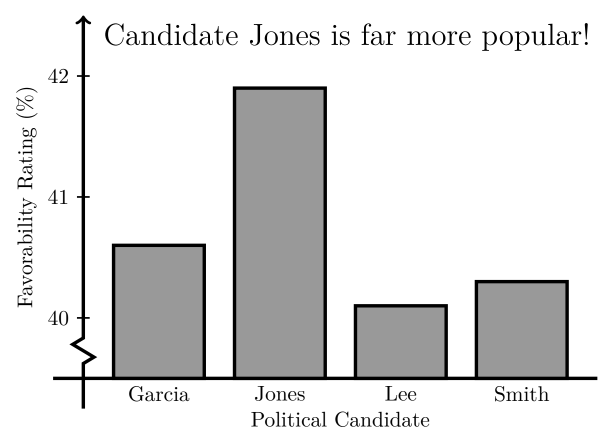
While they might look clever, using _________________ in data displays should always be ___________.



Example 5

This chart violates the ___________________ because the ___________ of the bars are not ________________ to their corresponding _________. Even though Jones does have the highest favorability, the difference in favorability _____________ to be much greater because the bar's ________ is much greater.

This occurs because the _________ on the __________ has been _____________.



A graph such as a line chart can also have a _____________________. In some cases, this is ____________ when seeing trends and small changes is important, such as in ______________________. In general, however, readers will expect a _________ scale beginning at ________.

