Algebra 2 Notes

Answer Key

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For Sarah, who proves every day that math equals love.

This resource is pre-release and is still a work in progress. All contents are subject to change. Use at your own risk.

Contents

1 Functions **5**
- 1.1 Sets . . . 6
- 1.2 Introduction to Functions . . . 9
- 1.3 Inverse Functions and Solving Equations . . . 12
- 1.4 Transformations . . . 15

2 Linear Functions and Equations **19**
- 2.1 Linear Functions . . . 20
- 2.2 Inverses of Linear Functions . . . 25
- 2.3 Systems of Linear Equations . . . 27
- 2.4 Linear Regression . . . 33
- 2.5 Piecewise Linear Functions . . . 36

3 Quadratic Functions and Equations **41**
- 3.1 Quadratics in Vertex Form . . . 42
- 3.2 Quadratics in Factored Form . . . 46
- 3.3 Review of Distributing and Factoring . . . 48
- 3.4 Special Quadratics . . . 52
- 3.5 Factoring Quadratics in Standard Form . . . 56
- 3.6 Completing the Square . . . 60
- 3.7 The Quadratic Formula . . . 63

4 Further Quadratics **67**
- 4.1 Complex Numbers . . . 68
- 4.2 Quadratic Equations with Complex Solutions . . . 72
- 4.3 Systems Involving Quadratic Equations . . . 74
- 4.4 Quadratic Regression . . . 77

5 Polynomials **79**
- 5.1 Polynomial Concepts . . . 80
- 5.2 Cubic Functions . . . 82
- 5.3 Special Cubics . . . 84
- 5.4 Polynomial Division . . . 86
- 5.5 Factoring Polynomials . . . 90
- 5.6 Graphs of Polynomial Functions . . . 92

6 Rational Expressions and Functions **97**

6.1 Simplifying Rational Expressions . . . 98
6.2 Adding and Subtracting Rational Expressions . . . 101
6.3 Complex Fractions . . . 104
6.4 Rational Equations . . . 106
6.5 Simple Rational Functions . . . 108
6.6 Functions with Quadratic Denominators . . . 112

7 Radicals and Rational Exponents 117
7.1 Radical Expression Concepts . . . 118
7.2 Rational Exponents . . . 121
7.3 Square Root Equations . . . 124
7.4 Square Root Functions . . . 126
7.5 Cube Root Functions . . . 130
7.6 Quadratics, Cubics and Roots as Inverses . . . 132

8 Exponential and Logarithmic Functions 135
8.1 Exponential Functions . . . 136
8.2 Logarithms . . . 139
8.3 Logarithmic Functions . . . 142
8.4 Natural Exponents and Logarithms . . . 145
8.5 Exponential and Logarithmic Equations . . . 149
8.6 Exponential Regression . . . 151

9 Further Functions 153
9.1 Identifying Functions . . . 154
9.2 Algebraic Combinations of Functions . . . 156
9.3 Function Composition . . . 158
9.4 Piecewise Functions . . . 161

10 Matrices 163
10.1 Matrix Operations . . . 164
10.2 Solving Linear Systems with Matrices . . . 166

11 Sequences and Series 169
11.1 Introduction to Sequences and Series . . . 170
11.2 Arithmetic Sequences and Series . . . 174
11.3 Geometric Sequences and Series . . . 176

12 Data and Statistics 179
12.1 Statistical Concepts . . . 180
12.2 Normal Distributions . . . 183
12.3 Bivariate Data . . . 185
12.4 Collecting and Presenting Data . . . 188

Chapter 1

Functions

1.1 Sets 6
1.2 Introduction to Functions 9
1.3 Inverse Functions and Solving Equations 12
1.4 Transformations 15

1.1 Sets

A set is a collection of mathematical objects. In this class, it will almost always be a collection of numbers. Sets are usually represented by upper case variables.

Sets can be defined as a list of values, or by using a rule, notated by curly braces.

Example 1 If set A contains only the values 1, 2, 3, 6, 8 and 9, then

$$A = \{1, 2, 3, 6, 8, 9\}$$

If set B contains all values greater than or equal to 6, then

$$B = \{x : x \geq 6\}$$

Note that either : or | can be used in set notation. If reading aloud, say "such that".

$x \in S$ says that the value x is an element of the set S, or x is in S.
$x \notin S$ says the opposite: the value x is not in the set S.

Example 2 Using the definitions of A and B above, write $\in$ or $\notin$.

$1 \in A$	$4 \notin A$	$6 \in A$	$7 \notin A$	$5.9 \notin A$	$8.1 \notin A$
$1 \notin B$	$4 \notin B$	$6 \in B$	$7 \in B$	$5.9 \notin B$	$8.1 \in B$

Symbols for Special Sets

Typed	Written	Name	Description
$\varnothing$		the empty set	The set that contains no elements at all.
$\mathbb{N}$		the natural numbers	The set of numbers[1] used for counting. $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{Z}$		the integers	The set containing all the natural numbers, their negative counterparts, and 0. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$		the rational numbers	The set of numbers which can be written as fractions using integers. Real numbers not in this set (including π) are called irrational.
$\mathbb{R}$		the real numbers	The set of all numbers which can be placed on the number line.

[1] Many mathematicians would say the natural numbers also include 0. If you want unambiguous terms, you can use *positive integers* to exclude 0, and *nonnegative integers* include 0.

Combining Sets

$A \cap B$ is the intersection of A and B. It is a set that contains all the elements that are in **both** A **and** B.

$A \cup B$ is the union of A and B. It is a set that contains all the elements that are in **either** A **or** B.

$A \backslash B$ is the set difference of A and B. It is a set that contains all the elements that are **in** A **but not in** B.

Example 3 $C = \{1, 5, 7, 10\}$ and $D = \{4, 5, 6, 7, 8\}$

$C \cap D = \{5, 7\}$ $\qquad$ $C \cup D = \{1, 4, 5, 6, 7, 8, 10\}$

$C \backslash D = \{1, 10\}$ $\qquad$ $D \backslash C = \{4, 6, 8\}$

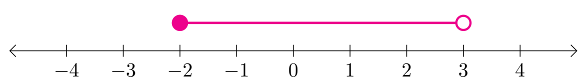
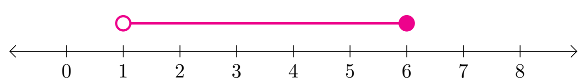
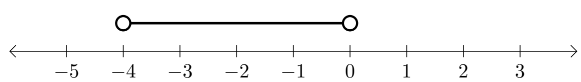
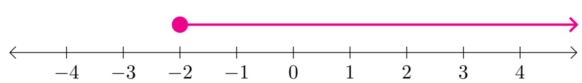
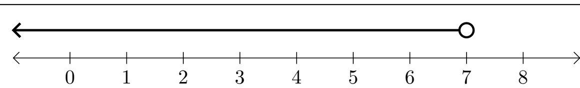
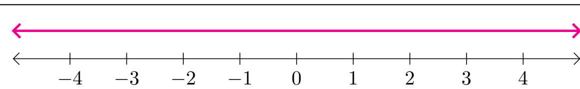
Interval Notation

An interval is a special type of set which contains all real numbers between a lower bound, a, and an upper bound, b.

$[a, b]$ represents an interval with bounds which are included. (a, b) represents an interval with bounds which are excluded. $(a, b]$ and $[a, b)$ can be used when the bound types are mixed.

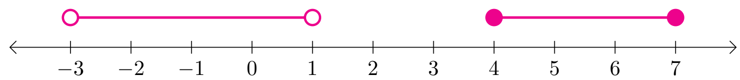
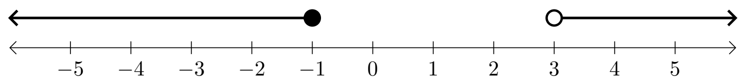
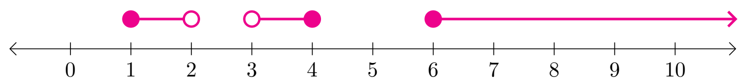
On number lines and graphs, an included bound is represented by a closed point, ●, and an excluded bound is represented by an open point, ○.

Example 4

Interval	Set Notation	Real Number Line
$[-2, 3)$	$\{x : -2 \leq x < 3\}$	
$(1, 6]$	$\{x \mid 1 < x \leq 6\}$	
$(-4, 0)$	$\{x \mid -4 < x < 0\}$	
$[-2, \infty)$	$\{x : x \geq -2\}$	
$(-\infty, 7)$	$\{x : x < 7\}$	
$(-\infty, \infty)$	$\mathbb{R} = \{x \mid x \text{ is real}\}$	

If a set consists of disconnected intervals, the union symbol can be used to include them in the same set.

Examples:

Interval Notation	Real Number Line
$(-3,1)\cup[4,7]$	
$(-\infty,-1]\cup(3,\infty)$	
$[1,2)\cup(3,4]\cup[6,\infty)$	

If a set contains all real numbers except some values, there are multiple options for notating the set.

Example 5 The set containing all real numbers except 2 and 5 is

Interval Notation	Set Notation	Set Difference
$(-\infty,2)\cup(2,5)\cup(5,\infty)$	$\{x \mid x\neq 2,5\}$	$\mathbb{R}\setminus\{2,5\}$

Comparing Sets

If every element in a set U is also in another set V, then we can write $U\subset V$. We say that U is a subset of V, and that V is a superset of U. We can also say that V contains U.

Example 6 Let $A=\{-1,2,3,4\}$ and $B=\{-1,2,3,4,5.5,7\}$.

Set Relation	T/F	Reason
$A\subset B$	True	Every number in A is also in B.
$B\subset A$	False	$7\in B$, but $7\notin A$.
$A\subset\mathbb{N}$	False	$-1\in A$, but -1 is not a natural number.
$A\subset\mathbb{Z}$	True	Every number in A is an integer.
$B\subset\mathbb{Z}$	False	$5.5\in B$, but 5.5 is not an integer.
$A\subset[-1,4)$	False	$4\in A$, but $4\notin[-1,4)$.
$B\subset[-1,7]$	True	Every number in B satisfies $-1\leq x\leq 7$.
$[-1,4)\subset[-1,7]$	True	If $-1\leq x<4$, then $-1\leq x\leq 7$ is also true.
$\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$	True	Follows from definitions of these sets.

1.2 Introduction to Functions

A relation is a collection of ordered pairs which represents a relationship between two sets of real numbers. Each ordered pair is typically labeled as (x, y).

The first set, which contains all x-values, is called the domain. The second set, which contains the y-values, is called the codomain.

A function is a particular type of relation. In a function, each value in the domain is uniquely related to a value in the codomain. In other words, for each x, there is exactly one y related to it.

To say that a function f relates a domain A and a codomain B, we write

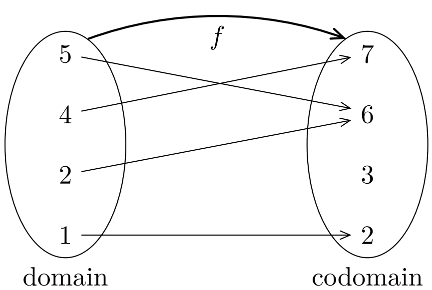
$$f : A \to B$$

which can be read aloud as f maps from A to B.

The relation between x and y is written as $y = f(x)$

The range (or image) of a function is the subset of the codomain that contains the values that are actually produced by the function. We can think of the domain as the inputs of the function, and the range as the outputs of the function.

Example 1 Find the domain, codomain and range of the function, and find the value of $f(x)$ for each value x in the domain.



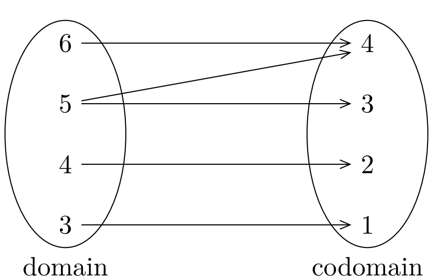
domain of $f = \{1, 2, 4, 5\}$

codomain of $f = \{2, 3, 6, 7\}$

range of $f = \{2, 6, 7\}$

$f(1) = 2 \quad f(2) = 6 \quad f(4) = 7 \quad f(5) = 6$

Example 2 Explain why the following relation is **not** a function.

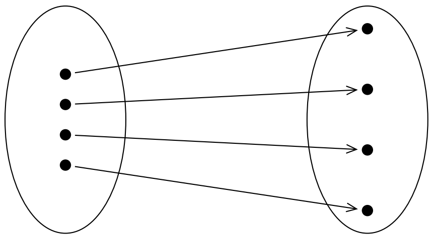


The value 5 in the domain maps to both 3 and 4 in the codomain.

As 5 is not uniquely related, this is not a function.

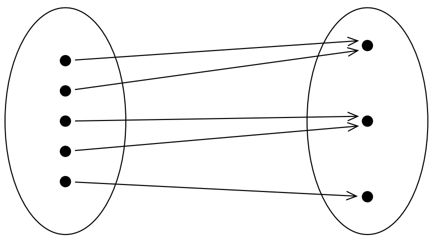
One-to-One and Many-to-One Functions

For every function, each x-value in the domain maps to a unique y-value in the range. It is not necessarily true that each y-value is mapped to by a unique x-value.



In a one-to-one function, each y-value in the range is only mapped to by one x-value in the domain.

Equivalently, $f(a) = f(b)$ if and only if $a = b$.



In a many-to-one function, at least one y-value in the range mapped to by more than one x-value in the domain.

Equivalently, there is an a and b in the domain such that $f(a) = f(b)$, but $a \neq b$.

Function Evaluation

To evaluate a function means to determine the value of $f(a)$ for a given value a in the domain. If a is not in the domain, then $f(a)$ is said to be undefined.

Example 3 The function f is defined by the table shown.

x	$f(x)$
-3	4
-2	3
-1	0
0	1
1	-1
2	5
3	2

The domain of f is $\{-3, -2, -1, 0, 1, 2, 3\}$.

The range of f is $\{-1, 0, 1, 2, 3, 4, 5\}$.

The relation type of f is one-to one, because each output has only one input.

$f(2) = 5$ $f(4)$ is undefined

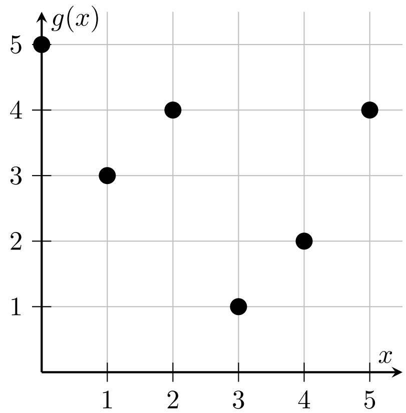
$f(-2) + f(2) = 3 + 5 = 8$

$2f(-3) - 5f(0) = 2 \cdot 4 - 5 \cdot 1 = 8 - 5 = 3$

$f(f(1)) = f(-1) = 0$

$f(f(f(-2))) = f(f(3)) = f(2) = 5$

Example 4 The function g is defined by the graph shown.



The domain of g is $\{0, 1, 2, 3, 4, 5\}$.

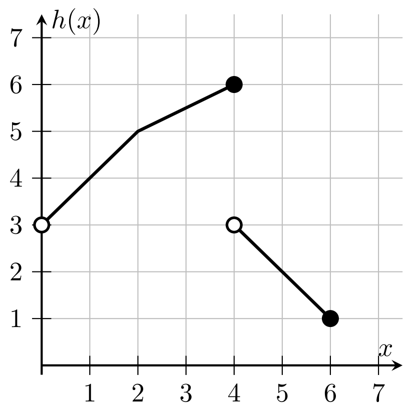
The range of g is $\{1, 2, 3, 4, 5\}$.

The relation type of g is many-to-one, because $g(2) = g(5) = 4$.

$g(3) = 1$ $g(1.5)$ is undefined

$g(g(g(0))) = g(g(5)) = g(4) = 2$

Example 5 The function h is defined by the graph shown.



The domain of h is $(0, 6]$.

The range of h is $[1, 3) \cup (3, 6]$.

The relation type of h is one-to-one, because each output has only one input.

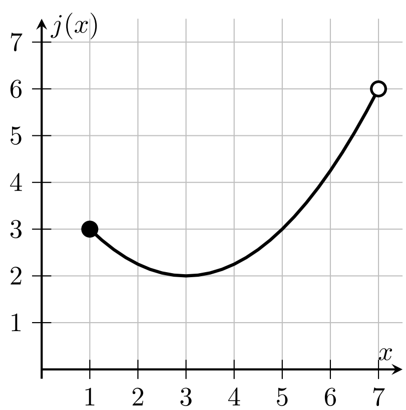
$h(4) = 6$ $h(1.5) = 4.5$

$h(0)$ is undefined $h(2.5) = 5.25$

$h(g(1)) = h(3) = 5.5$

$g(h(1)) = g(4) = 2$

Example 6 The function j is defined by the graph shown.



The domain of j is $[1, 7)$.

The range of j is $[2, 6)$.

The relation type of j is many-to-one, because $j(1) = j(5) = 3$.

$j(3) = 2$ $j(7)$ is undefined

$j(2) = 2.25$ $j(6) = 4.25$

1.3 Inverse Functions and Solving Equations

Suppose we have a relation, which consists of a collection of ordered pairs in the form (x, y). Its inverse relation is the relation whose ordered pairs are switched to be (y, x).

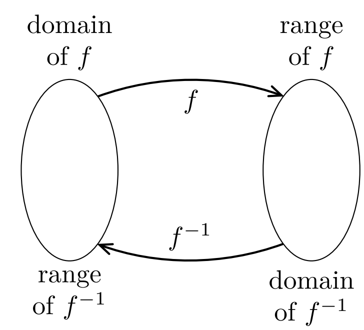
Recall that a function is a special type of relation. If the inverse relation of a function is also a function, it is called the inverse function.

If a function is denoted f, its inverse function, if it exists, is denoted f^{-1}.

Properties of Inverse Functions

If function f has the inverse function f^{-1}, then

- The inverse function of f^{-1} is f.
- The domain of f^{-1} is identical to the range of f.
- The range of f^{-1} is identical to the domain of f.
- As the inverse function results from switching the x and y values, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections, or mirror images of each other across the line y = x.



Condition for Inverse Functions

Suppose function f is defined by the following table, and suppose f^{-1} is its inverse function.

x	1	2	3
$f(x)$	7	8	7

What is $f^{-1}(8)$? $f^{-1}(8) = 2$ because $f(2) = 8$.

What is $f^{-1}(7)$? $f^{-1}(7) = 1$ or $f^{-1}(7) = 3$ because $f(1) = f(3) = 7$.

Because $f^{-1}(7)$ has multiple values, f^{-1} is not a function. This has happened because f is a many-to-one function. Therefore,

Theorem

A function f has an inverse function f^{-1} if and only if f is a one-to-one function.

Example 1 The function f is defined by the table shown.

x	$f(x)$
-3	4
-2	3
-1	0
0	1
1	-1
2	2

The domain of f is $\{-3, -2, -1, 0, 1, 2\}$.

The range of f is $\{-1, 0, 1, 2, 3, 4\}$.

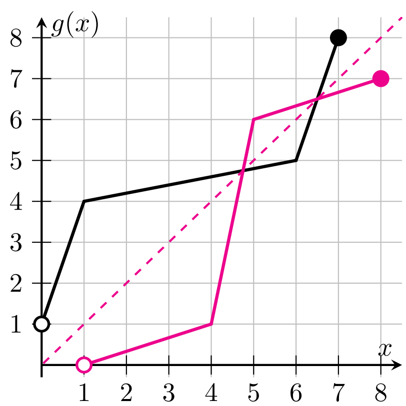
The inverse function f^{-1} does exist because the function is one-to one.

The domain of f^{-1} is $\{-1, 0, 1, 2, 3, 4\}$.

The range of f^{-1} is $\{-3, -2, -1, 0, 1, 2\}$.

x	$f^{-1}(x)$
-1	1
0	-1
1	0
2	2
3	-2
4	-3

Example 2 The function g is defined by the graph shown.



The domain of g is $(0, 7]$.

The range of g is $(1, 8]$.

The inverse function g^{-1} does exist because the function is one-to one.

The domain of g^{-1} is $(1, 8]$.

The range of g^{-1} is $(0, 7]$.

$g(1) = 4$ $g(6) = 5$ $g(7) = 8$

$g^{-1}(4) = 1$ $g^{-1}(5) = 6$ $g^{-1}(8) = 7$

Solving Equations using Inverse Functions

Recall that we can use inverse operations to solve equations. If an equation contains a one-to-one function, we can use its inverse function in the same way to solve the equation.

If a solution exists, this method will ensure that it is unique. If the equation requires applying the inverse function to a value for which it is undefined, then the equation has no solution.

Example 3 Solve the following equations using the table defining f.

x	-3	-2	-1	0	1	2	3
$f(x)$	4	3	0	1	-1	5	2

$2f(x+3)-4=6$

$$\begin{aligned} 2f(x+3) &= 6+4 \\ &= 10 \\ f(x+3) &= \tfrac{10}{2} \\ &= 5 \\ x+3 &= f^{-1}(5) \\ &= 2 \\ x &= 2-3 \\ &= -1 \end{aligned}$$

$\dfrac{f(5x)-1}{3}=2$

$$\begin{aligned} f(5x)-1 &= 2\cdot 3 \\ &= 6 \\ f(5x) &= 6+1 \\ &= 7 \\ 5x &= f^{-1}(7) \end{aligned}$$

is undefined

$\therefore$ no solution

Solving Equations with no Inverse Function

If an equation contains a many-to-one function, it may still be possible to solve the equation. However, the solution may not be unique.

Example 4 Solve the following equations using the table defining g.

x	-3	-2	-1	0	1	2	3
$g(x)$	3	2	1	3	2	1	3

$3g(x-5)+2=8$

$$\begin{aligned} 3g(x-5) &= 8-2 \\ &= 6 \\ g(x-5) &= \frac{6}{3} \\ &= 2 \end{aligned}$$

$x-5=-2$ or $x-5=1$

$x=-2+5$ or $x=1+5$

$x=3$ or $x=6$

$\dfrac{g(x)+7}{2}=5$

$$\begin{aligned} g(x)+7 &= 5\cdot 2 \\ &= 10 \\ g(x) &= 10-7 \\ &= 3 \end{aligned}$$

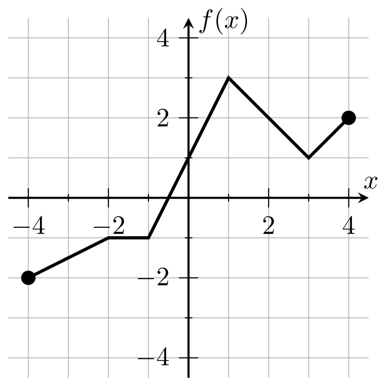
$x=-3$ or $x=0$ or $x=3$

1.4 Transformations

A transformation is a rule which, when applied to a geometric figure, produces an image of the figure with each point changed in a prescribed way.

In this class we'll consider transformations of graphs of functions and how they change the function algebraically.

For the following examples, we'll use the function f, as defined by this graph and table:



x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1.5	-1	-1	1	3	2	1	2

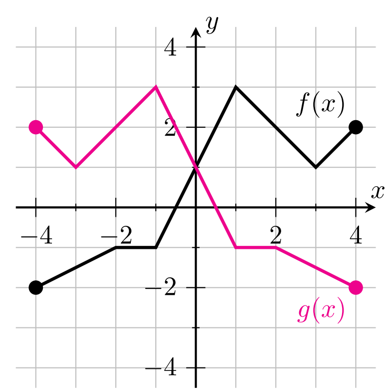
Reflections

A reflection is a transformation which creates a mirror image across a line of symmetry. Each point in the image remains the same distance from this line, but on the opposite side.

Example 1

$$g(x) = f(-x)$$

x	4	3	2	1	0	-1	-2	-3	-4
$-x$	-4	-3	-2	-1	0	1	2	3	4
$f(-x)$ $g(x)$	-2	-1.5	-1	-1	1	3	2	1	2



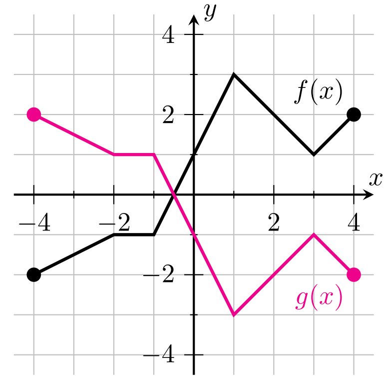
Each x-value has the opposite sign. Each y-value is unchanged.

The graph has been reflected across the y-axis.

Example 2

$$g(x) = -f(x)$$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1.5	-1	-1	1	3	2	1	2
$-f(x)$ $g(x)$	2	1.5	1	1	-1	-3	-2	-1	-2



Each x-value is unchanged. Each y-value has the opposite sign.

The graph has been reflected across the x-axis.

Stretches and Compressions

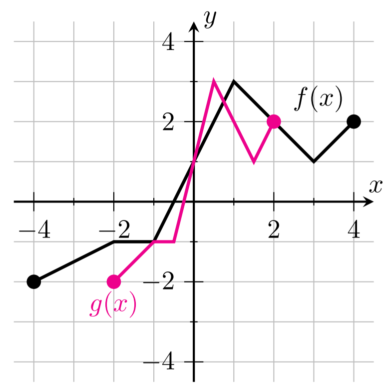
A stretch or compression is a transformation where each point's distance from a fixed line is multiplied by a scale factor.

If each point gets further from the fixed line, the transformation is a stretch. If each point gets closer to the fixed line, the transformation is a compression.

Example 3

$$g(x) = f(2x)$$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$2x$	-4	-3	-2	-1	0	1	2	3	4
$f(2x)$ $g(x)$	-2	-1.5	-1	-1	1	3	2	1	2



Each x-value is multiplied by ½. Each y-value is unchanged.

The graph has been compressed toward the y-axis by a factor of 2.

Example 4

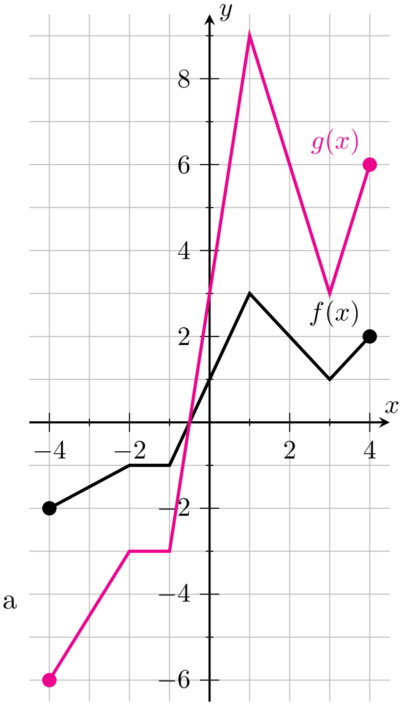
$$g(x) = 3f(x)$$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1.5	-1	-1	1	3	2	1	2
$3f(x)$ $g(x)$	-6	-4.5	-3	-3	3	9	6	3	6

Each x-value is unchanged.

Each y-value is multiplied by 3.

The graph has been stretched from the x-axis by a factor of 3.



Translations

A translation, or shift, is a transformation where every point in the image is moved the same distance in the same direction.

A translation can be left or right, or up or down, or a combination of directions.

Example 5

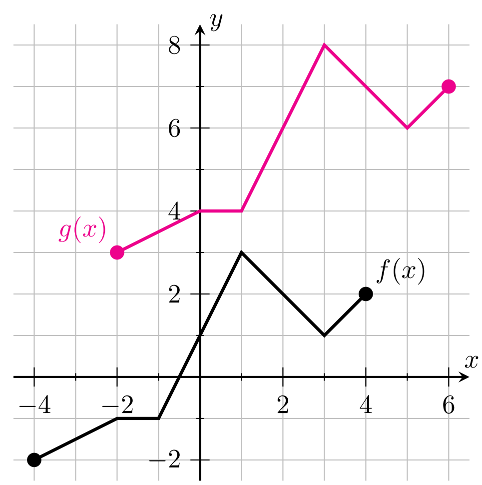
$$g(x) = f(x-2) + 5$$

x	-2	-1	0	1	2	3	4	5	6
$x-2$	-4	-3	-2	-1	0	1	2	3	4
$f(x-2)$	-2	-1.5	-1	-1	1	3	2	1	2
$f(x-2)+5$ $g(x)$	3	3.5	4	4	6	8	7	6	7

Each x-value is increased by 2.

Each y-value is increased by 5.

The graph has been shifted 2 units right and 5 units up.



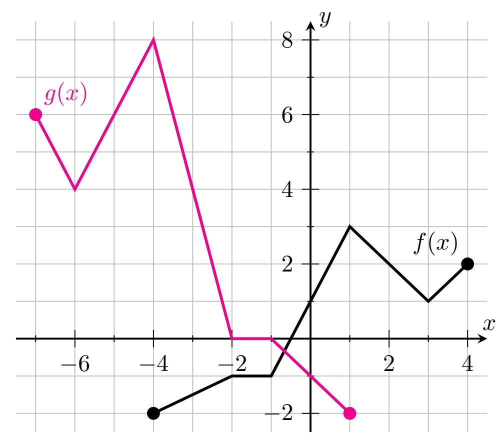
Combining Transformations

Example 6 $$g(x) = 2f\left[-(x+3)\right] + 2$$

x	1	0	-1	-2	-3	-4	-5	-6	-7
$x+3$	4	3	2	1	0	-1	-2	-3	-4
$-(x+3)$	-4	-3	-2	-1	0	1	2	3	4
$f\left[-(x+3)\right]$	-2	-1.5	-1	-1	1	3	2	1	2
$2f\left[-(x+3)\right]$	-4	-3	-2	-2	2	6	4	2	4
$2f\left[-(x+3)\right]+2$ $g(x)$	-2	-1	0	0	4	8	6	4	6

The graph has been:

- reflected across the y-axis,
- stretched from the x-axis by a factor of 2,
- shifted left by 3 units, and
- shifted up by 2 units.



When listing transformations for the usual form $g(x) = A \cdot f\left[n(x-h)\right] + k$, translations should always be listed after reflections and dilations.

Summary of Transformations

$y = A \cdot f(x)$	reflect across the x-axis if A is negative stretch from the x-axis by a factor of $\lvert A\rvert$ if $\lvert A\rvert > 1$ compress toward the x-axis by a factor of $\frac{1}{\lvert A\rvert}$ if $0 < \lvert A\rvert < 1$
$y = f(n \cdot x)$	reflect across the y-axis if n is negative stretch from the y-axis by a factor of $\frac{1}{\lvert n\rvert}$ if $0 < \lvert n\rvert < 1$ compress toward the y-axis by a factor of $\lvert n\rvert$ if $\lvert n\rvert > 1$
$y = f(x-h) + k$	translate $\lvert h\rvert$ units right if h is positive, left if negative translate $\lvert k\rvert$ units up if k is positive, down if negative

Chapter 2

Linear Functions and Equations

2.1 Linear Functions 20
2.2 Inverses of Linear Functions 25
2.3 Systems of Linear Equations 27
2.4 Linear Regression 33
2.5 Piecewise Linear Functions 36

2.1 Linear Functions

A linear function is a function with the algebraic form

$$f(x) = mx + b$$

where m and b are constants.

This corresponds to the slope-intercept form of a linear relation, named because the graph of the function is a straight line, where m is the slope of the line and b is its y-intercept.

If a function is defined by an equation rule, the function is evaluated by substituting the appropriate value from the domain into the rule, and calculating the result.

Example 1 $f : [-3, 6) \to \mathbb{R}$, where $f(x) = -2x + 8$.

$$f(2) = -2(2) + 8 \\ = 4$$

$$f(5) = -2(5) + 8 \\ = -2$$

$$f(-3) = -2(-3) + 8 \\ = 14$$

$f(7)$ is undefined $\because 7 \notin [-3, 6)$

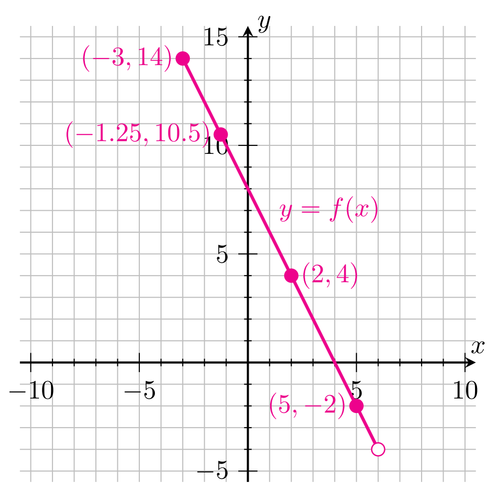
$$f(-1.25) = -2(-1.25) + 8 \\ = 2.5 + 8 \\ = 10.5$$

Graphing Functions

A useful tool to visualize a function is its graph. The graph consists of a curve[1] drawn on a coordinate plane, or Cartesian plane[2].

If x is in the domain of the function f, then the point $(x, f(x))$ will be part of the curve.

Example 2 Plot the function f from Example 1 on the coordinate plane to the right.



[1] Even if it's a straight line, it's still called a "curve".
[2] Named after the 17th Century French philosopher, René Descartes.

Implied Domains

It is common practice to state only the rule of a function, without stating the domain. In these cases, it is reasonable to assume the implied domain, which is the largest possible domain for which the function can be evaluated.

For a linear function, the implied domain is all real numbers, $\mathbb{R}$, because

$mx + b$ can be evaluated for any $x \in \mathbb{R}$.

Sketching Linear Functions

A sketch is a version of a graph that shows only the key information. In the case of a linear function, the information that should be included is:

shape of curve	straight line with an appropriate slope
x-intercept	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$
endpoints	evaluate the function at the bounds of the domain

Example 3 Sketch $f(x) = 4x + 6$.

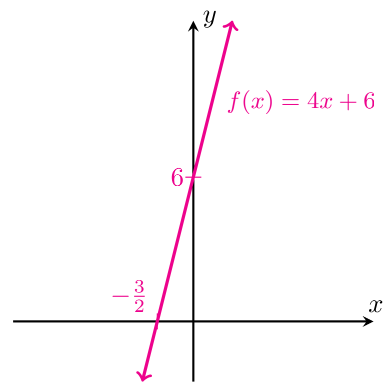
Shape: Straight line with slope $m = 4$

x-intercept: $\left(-\frac{3}{2}, 0\right)$

$$4x + 6 = 0$$
$$4x = -6$$
$$x = -\tfrac{3}{2}$$

y-intercept: $(0, 6)$, as $f(0) = 6$

endpoints: none, as domain is $\mathbb{R}$



Example 4 Sketch $g(x) = -\frac{1}{2}x + 1$ on the domain $[2, \infty)$.

Shape: Straight line with slope $m = -\frac{1}{2}$

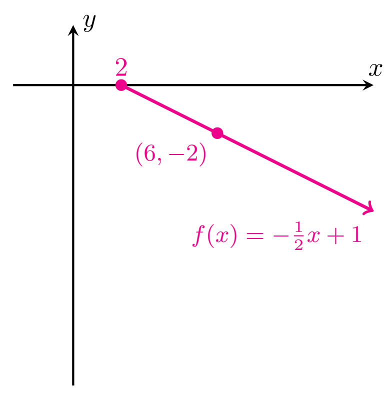
x-intercept: $(2, 0)$

$$-\tfrac{1}{2}x + 1 = 0$$
$$-\tfrac{1}{2}x = -1$$
$$x = 2$$

y-intercept: none, as $f(0)$ is undefined

endpoints: $(2, 0)$

$g(2) = 0$



Note that it is a good idea to include at least two points so the slope of the line is clear.

Example 5 Find the range of $h : (-1, 5] \to \mathbb{R}$ where $h(x) = -2x - 3$, and sketch the graph of $h(x)$.

Shape: Straight line with slope $m = -2$

x-intercept: none, as $x = -\frac{3}{2} \notin (-1, 5]$

$-2x - 3 = 0$

$-2x = 3$

$x = -\frac{3}{2}$

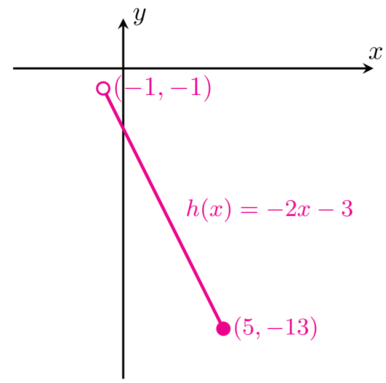
y-intercept: $(0, -3)$, as $h(0) = -3$

endpoints: $(-1, -1)$ and $(-5, -13)$

$x = -1 :\quad -2(-1) - 3 = 2 - 3 = -1$

$x = 5 :\quad h(5) = -2(5) - 3 = -10 - 3 = -13$

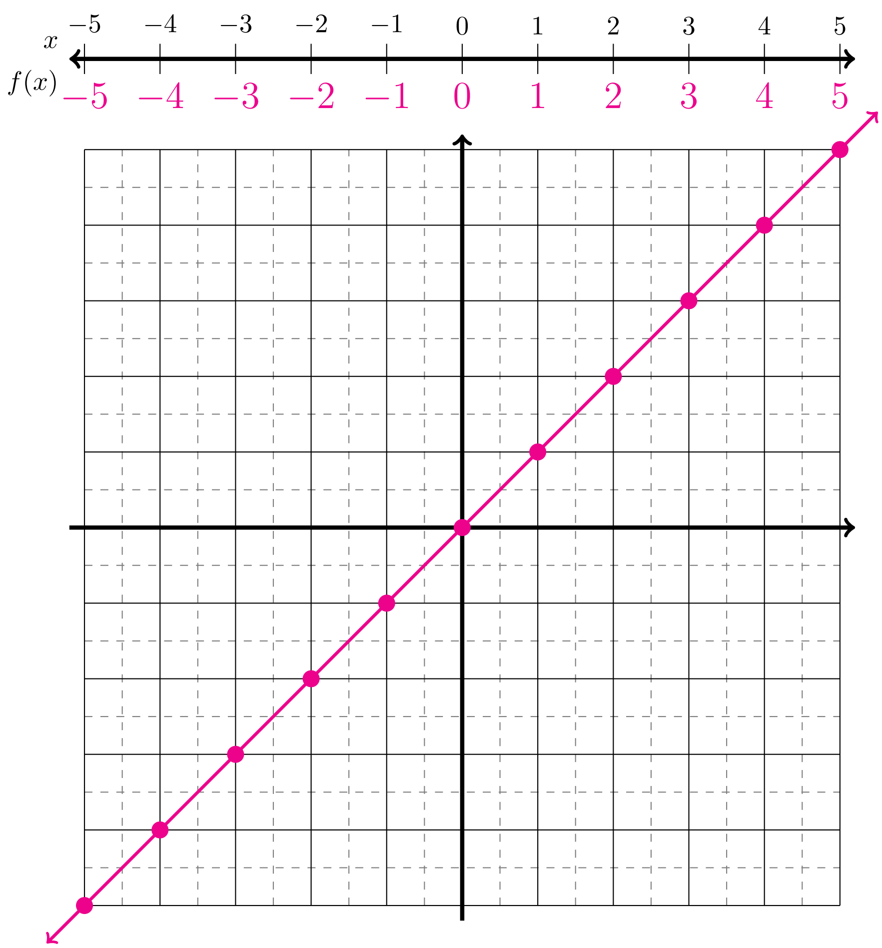
The range of h is $[-13, -1)$



The Linear Parent Function

For any given function, its parent function is the simplest function of the same type.

parent function $f(x) = x$
domain $\mathbb{R}$
range $\mathbb{R}$
relation type one-to-one
x-intercept $(0, 0)$
y-intercept $(0, 0)$
slope 1



Transformations of Linear Functions

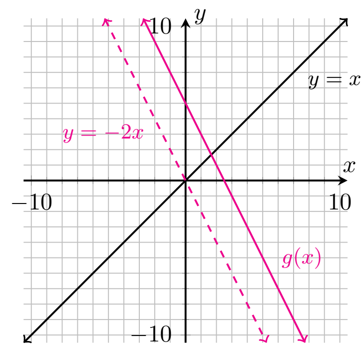
Recall that $g(x) = Af(x) + k$ represents a stretch or compression from the x-axis if $|A| \neq 1$, a reflection across the x-axis if A is negative, and a translation up or down.

If we let $A = m$, $k = b$, and $f(x) = x$, then $g(x) = mx + b$, the general form of linear functions. This gives us the following result:

Theorem

Every linear function, $g(x) = mx + b$, is the result of a transformation applied to its parent function, $f(x) = x$.

Example 6 Write the transformations needed to obtain $g(x) = -2x + 5$ from its parent function.



- Reflect across the x-axis.
- Stretch from the x-axis by a factor of 2.
- Shift 5 units up.

Example 7 The graph of $y = x$ is compressed by a factor of 4 toward the x-axis, shifted 8 units left and shifted 7 units down. What is resulting function in slope-intercept form?

$$A = \tfrac{1}{4}, \quad h = -8, \quad k = -7$$

$$\begin{aligned} f(x) &= \tfrac{1}{4}(x+8) - 7 \\ &= \tfrac{1}{4}x + 2 - 7 \\ &= \tfrac{1}{4}x - 5 \end{aligned}$$

Transformations do not need to be applied only to the parent function, but can be used with any function.

Example 8 The function $f : [-2, 5) \to \mathbb{R}$, where $f(x) = 2x + 4$, is reflected across the x-axis and shifted 3 units right. Find the resulting function g in the form $g(x) = mx + b$.

Find the new domain:

Reflecting across the x-axis does not affect the x values.

Shifting 3 units right means each x value is increased by 3.

So, $g : [1, 8) \to \mathbb{R}$.

Find the new rule:

$$\begin{aligned} g(x) &= -f(x-3) \\ &= -\left[2\left(x-3\right)+4\right] \\ &= -(2x-6+4) \\ &= -(2x-2) \\ &= -2x+2 \end{aligned}$$

Example 9 Find the transformations required to transform $f(x) = 3x + 2$ to $g(x) = -6x + 5$.

$$\begin{aligned} g(x) &= -6x+5 \\ &= -2(3x)+5 \\ &= -2(3x+2-2)+5 \\ &= -2(3x+2)+4+5 \\ &= -2f(x)+9 \end{aligned}$$

$A = -2, \quad k = 9$

- Reflect across the x-axis.
- Stretch from the x-axis by a factor of 2.
- Shift 9 units up.

2.2 Inverses of Linear Functions

Recall that a function has an inverse function if and only if it is a one-to-one function.

Since non-constant linear functions are one-to-one (think about why this is true) we can conclude the following:

Theorem

Each linear function, $f(x) = mx + b$, where $m \neq 0$, has an inverse function.

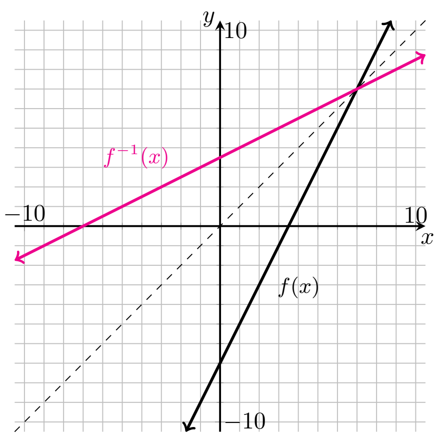
Finding the Inverse Function

Recall that the inverse of a relation results from reversing ordered pairs. For an algebraically defined function, we can find the inverse by following these steps:

1. Replace $f(x)$ with y.
2. Rewrite the equation by swapping x and y.
3. Rearrange the equation so that y is isolated.
4. Check that y is a function; if so, replace y with $f^{-1}(x)$.

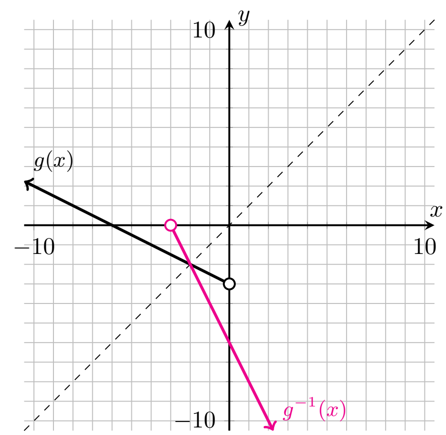
Example 1 Find the inverse function of $f(x) = 2x - 7$.

$$y = 2x - 7$$
$$x = 2y - 7 \qquad \text{swap } x \leftrightarrow y$$
$$2y = x + 7$$
$$y = \tfrac{1}{2}x + \tfrac{7}{2}$$
$$f^{-1}(x) = \tfrac{1}{2}x + \tfrac{7}{2}$$



Example 2 Find the inverse of $g : (-\infty, 0) \to \mathbb{R}$, where $g(x) = -\frac{1}{2}x - 3$.

$$\begin{aligned} y &= -\tfrac{1}{2}x - 3 \\ x &= -\tfrac{1}{2}y - 3 \qquad \text{swap } x \leftrightarrow y \\ -\tfrac{1}{2}y &= x + 3 \\ y &= -2x - 6 \\ g^{-1}(x) &= -2x - 6 \end{aligned}$$



We also need to find the domain of g^{-1}, which is the same as the range of g:

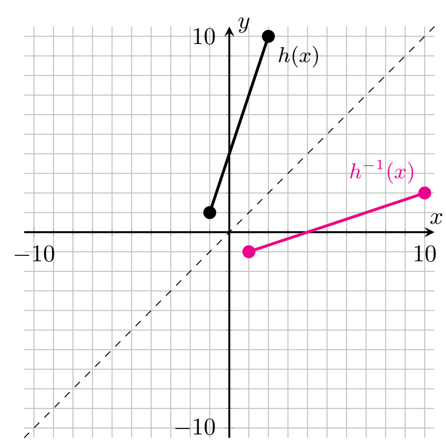
$$\begin{aligned} x &< 0 \\ -\tfrac{1}{2}y &> 0 \\ -\tfrac{1}{2}y - 3 &> -3 \\ g(x) &> -3 \end{aligned}$$

domain of g^{-1} = range of $g = (-3, \infty)$

$\therefore g^{-1} : (-3, \infty) \to \mathbb{R}$, where $g^{-1}(x) = -2x - 6$

Example 3 Find the inverse of $h : [-1, 2] \to \mathbb{R}$, where $h(x) = 3x + 4$.

$$\begin{aligned} y &= 3x + 4 \\ x &= 3y + 4 \qquad \text{swap } x \leftrightarrow y \\ 3y &= x - 4 \\ y &= \tfrac{1}{3}x - \tfrac{4}{3} \\ h^{-1}(x) &= \tfrac{1}{3}x - \tfrac{4}{3} \end{aligned}$$



In the domain of h,

$$\begin{aligned} -1 \le\ & x \le 2 \\ -3 \le\ & 3x \le 6 \\ 1 \le\ & 3x + 4 \le 10 \\ 1 \le\ & h(x) \le 10 \end{aligned}$$

domain of h^{-1} = range of $h = [1, 10]$

$\therefore h^{-1} : [1, 10] \to \mathbb{R}$, where $h^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$

2.3 Systems of Linear Equations

A system of equations is a collection of multiple equations containing multiple unknowns, or variables. A solution to the system consists of values for the unknowns that satisfy all of the equations simultaneously.

Example 1 Verify that $x = 2$, $y = 5$, $z = -3$ is a solution to

$$\begin{cases} x + y + z = 4 \\ 2x - y - z = 2 \\ x + 3y + 2z = 11 \end{cases}$$

$$\begin{aligned} &x + y + z \\ &= 2 + 5 + (-3) \\ &= 4 \end{aligned}$$

$$\begin{aligned} &2x - y - z \\ &= 2(2) - 5 - (-3) \\ &= 4 - 5 + 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} &x + 3y + 2z \\ &= 2 + 3(5) + 2(-3) \\ &= 2 + 15 - 6 \\ &= 11 \end{aligned}$$

Solving Systems of Two Equations Using Substitution

1. Choose one equation, and rearrange it to isolate one unknown.
2. Substitute this equation into the other and solve for the remaining unknown.
3. Substitute this solution into the first rearranged equation to find the first unknown.
4. State the final solution for both unknowns, by stating each value separately or together as an ordered pair.

Example 2
$$\begin{cases} x + 2y = 10 & (1) \\ 2x - 3y = 6 & (2) \end{cases}$$

Rearrange (1): $$x = 10 - 2y \quad (3)$$

Sub into (2):
$$\begin{aligned} 2(10 - 2y) - 3y &= 6 \\ 20 - 4y - 3y &= 6 \\ -7y &= -14 \\ y &= 2 \end{aligned}$$

Sub into (3): $$x = 10 - 2(2) = 6$$

Solution: $$x = 6, \quad y = 2$$

Example 3 $\begin{cases} 2x - 3y = -11 & (1) \\ 3x - y = 8 & (2) \end{cases}$

Rearrange (2): $y = 3x - 8$ (3)

Sub into (1):
$$\begin{aligned} 2x - 3(3x - 8) &= -11 \\ 2x - 9x + 24 &= -11 \\ -7x &= -35 \\ x &= 5 \end{aligned}$$

Sub into (3): $y = 3(5) - 8 = 7$

Solution: $x = 5, \quad y = 7$

Solving Systems of Two Equations Using Elimination

1. Choose one unknown you want to have opposite coefficients. Make this true by multiplying the equations by appropriate values.
2. Eliminate this unknown by adding the equations.
3. Solve for the remaining unknown.
4. Substitute this solution into one of the original equations to find the first unknown.
5. State the final solution for both unknowns.

Example 4 $\begin{cases} 4x + 5y = -5 & (1) \\ -2x - y = 7 & (2) \end{cases}$

Multiply (2) by 5:

$$\begin{cases} 4x + 5y = -5 & (3) \\ -10x - 5y = 35 & (4) \end{cases}$$

Add (3) and (4):

$$\begin{aligned} -6x &= 30 \\ x &= -5 \end{aligned}$$

Sub into (2):

$$\begin{aligned} -2(-5) - y &= 7 \\ 10 - y &= 7 \\ y &= 3 \end{aligned}$$

Solution:

$x = -5, \quad y = 3$

Example 5
$$\begin{cases} 3x + 4y = 2 & (1) \\ 2x - 5y = 9 & (2) \end{cases}$$

Multiply (1) by 2 and (2) by -3:

$$\begin{cases} 6x + 8y = 4 & (3) \\ -6x + 15y = -27 & (4) \end{cases}$$

Add (3) and (4):

$$23y = -23$$
$$y = -1$$

Sub into (1):

$$3x + 4(-1) = 2$$
$$3x - 4 = 2$$
$$3x = 6$$
$$x = 2$$

Solution:

$$x = 2, \quad y = -1$$

Solving Systems of Two Equations Using Graphs

Recall that when an equation is graphed, each point on the curve represents an ordered pair that satisfies the equation.

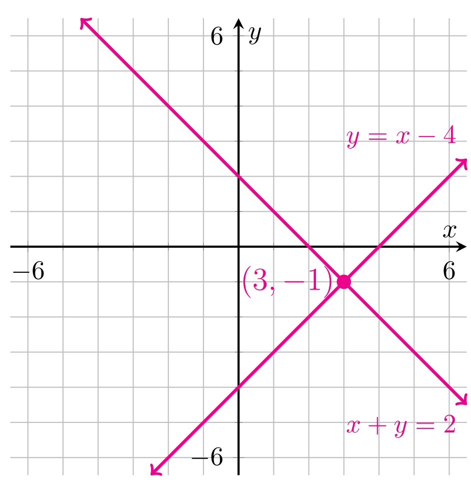
Suppose both equations of a system are graphed on the same plane. Any points of intersection will represent ordered pairs which satisfy both equations. This is exactly what we're looking for as a solution to the system.

Example 6

$$\begin{cases} y = x - 4 & (1) \\ x + y = 2 & (2) \end{cases}$$

$(2) \implies y = -x + 2$

Solution at $x = 3$, $y = -1$

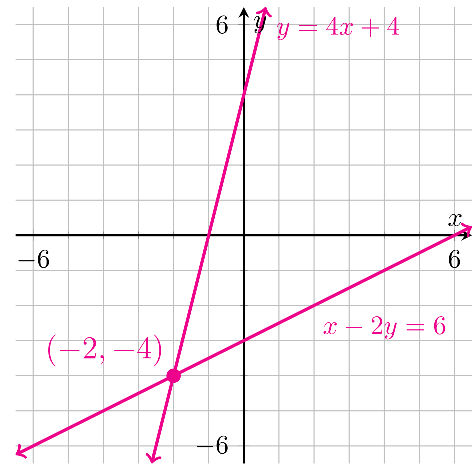


Example 7

$$\begin{cases} x - 2y = 6 & (1) \\ \quad\;\; y = 4x + 4 & (2) \end{cases}$$

(1) $\implies y = \frac{1}{2}x - 3$

Solution at $x = -2,\ y = -4$



Types of Solutions to Systems of Linear Equations

Each of the earlier example systems have <u>exactly one solution</u>. This is not always the case. Linear systems may instead have <u>infinitely many solutions</u>, or have <u>no solution</u>.

Example 8 Algebraically find the nature of the solution to this system. Represent it with a graph.

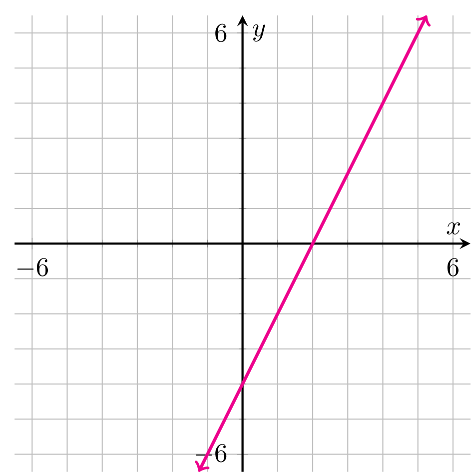
$$\begin{cases} 2x - \;\; y = 4 & (1) \\ 6x - 3y = 12 & (2) \end{cases}$$

Multiply (1) by -3:

$$\begin{cases} -6x + 3y = -12 & (3) \\ \;\; 6x - 3y = 12 & (4) \end{cases}$$

Add (3) and (4): $0 = 0$

$\implies$ infinitely many solutions



These equations are <u>equivalent</u>, because they <u>are always true</u> at the same time. The graphical representation has <u>infinitely many intersections</u> because the lines are <u>coincident</u>.

Example 9 Algebraically find the nature of the solution to this system. Represent it with a graph.

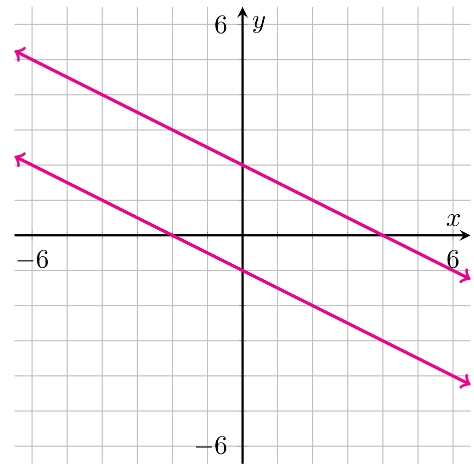
$$\begin{cases} x + 2y = -2 & (1) \\ 2x + 4y = 8 & (2) \end{cases}$$

Multiply (1) by -2:

$$\begin{cases} -2x - 4y = 4 & (3) \\ 2x + 4y = 8 & (4) \end{cases}$$

Add (3) and (4): $0 = 12$

$\implies$ no solution



These equations are inconsistent, because they cannot be true at the same time. The graphical representation has no intersection because the lines are parallel.

Systems of Three Linear Equations

For a system of three equations with three unknowns, we can use the same techniques to find a solution.

1. Use substitution or elimination to remove one unknown from the system.
2. Solve for the remaining two unknowns.
3. Use the partial solution to solve for the removed unknown. State the complete solution.

Example 10 Using substitution:

$$\begin{cases} x + y + z = 6 & (1) \\ 2x - y + 3z = 11 & (2) \\ -x + 3y + 4z = 8 & (3) \end{cases}$$

Rearrange (1): $x = -y - z + 6$ (4)

Sub (4) into (2):

$$2(-y - z + 6) - y + 3z = 11$$
$$-2y - 2z + 12 - y + 3z = 11$$
$$z = 3y - 1 \qquad (5)$$

Sub (4) into (3):

$$-(-y - z + 6) + 3y + 4z = 8$$
$$y + z - 6 + 3y + 4z = 8$$
$$4y + 5z = 14 \qquad (6)$$

Sub (5) into (6):

$$4y + 5(3y - 1) = 14$$
$$4y + 15y - 5 = 14$$
$$19y = 19$$
$$y = 1$$

Sub into (5): $z = 3(1) - 1 = 2$

Sub into (4): $x = -1 - 2 + 6 = 3$

$$x = 3, \ y = 1, \ z = 2$$

Example 11 Using elimination:

$$\begin{cases} x + y + z = 6 & (1) \\ 2x - y + 3z = 11 & (2) \\ -x + 3y + 4z = 8 & (3) \end{cases}$$

Add (1) + (2):

$$3x + 4z = 17 \qquad (4)$$

Add 3(2) + (3):

$$5x + 13z = 41 \qquad (5)$$

Multiply (4) by 5 and (5) by -3:

$$\begin{cases} 15x + 20z = 85 & (6) \\ -15x - 39z = -123 & (7) \end{cases}$$

Add (6) + (7):

$$-19z = -38$$

$$z = 2$$

Sub into (4):

$$3x + 4(2) = 17$$

$$3x + 8 = 17$$

$$3x = 9$$

$$x = 3$$

Sub into (1):

$$3 + y + 2 = 6$$

$$y = 1$$

$$x = 3,\ y = 1,\ z = 2$$

2.4 Linear Regression

Functions are often used for modeling real-world situations. Typically, the value of an independent variable is used as an input for the function, whose output is used to predict the value of a dependent variable.

Scatter Plots

A scatter plot is a plot used to visualize the relationship between two-variables, where each data point is treated as an ordered pair and plotted as a point on a plane.

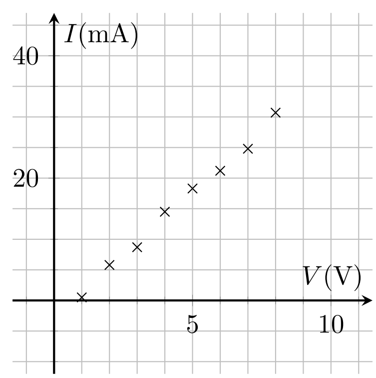
Visually inspecting a scatter plot can help decide whether a linear function is an appropriate model for a given set of data.

The independent variable is placed on the horizontal axis, and the dependent variable is placed on the vertical axis.

Example 1 A voltage source is placed in an electronic circuit. For various voltages, the current in the circuit is measured. The following results are recorded:

V (V)	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
I (mA)	0.5	5.8	8.7	14.5	18.3	21.2	24.8	30.7

Note that voltage, V, is measured in volts, V, and current, I, is measured in milliampere, mA.



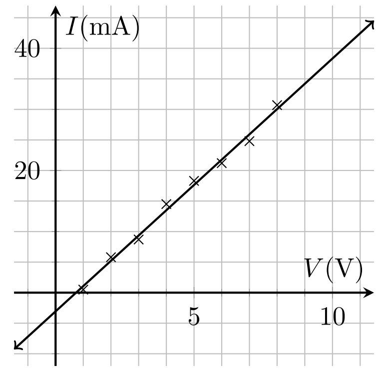
Regression

The process of fitting a function to a set of data in order to approximate the association between variables is called regression. When the modeling function is linear, it is called linear regression.

Since a linear function has the form $f(x) = mx + b$, linear regression means choosing values for m and b in order to fit the data as well as possible.[3] We will be using technology to find these values for us.

[3]You may think "as well as possible" is very vague. If so, you're right! The details of what this means are not important for Algebra 2, but they will be *very* important if you take a Statistics class in the future.

Example 2 For the electronic circuit example,

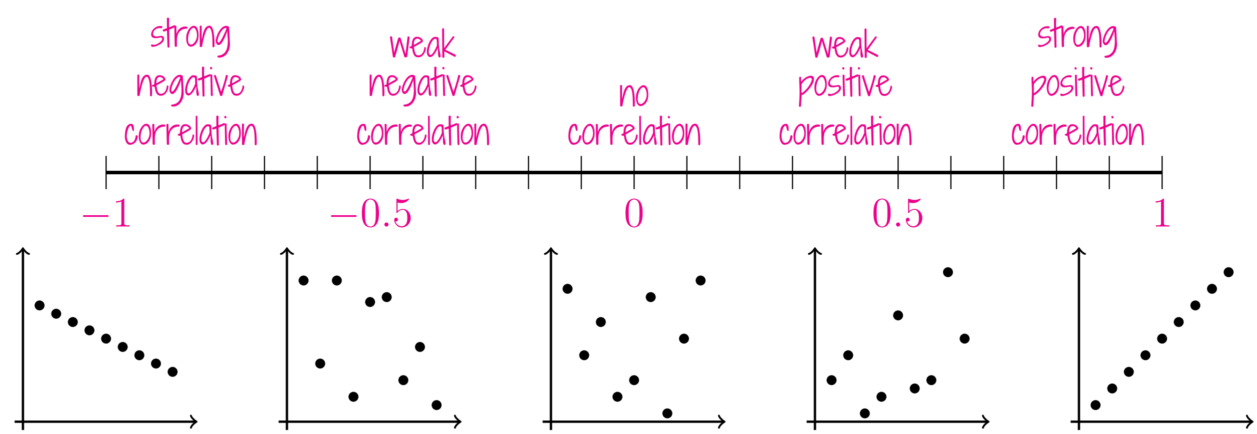


$$m = 4.13929, \quad b = -3.06429$$

$$I \approx 4.139V - 3.064$$

The Correlation Coefficient

The correlation coefficient, denoted by r, is a quantity that measures the strength and direction of the linear association between two variables. r is in the interval $[-1, 1]$.



Example 3 For the electronic circuit example, $r = 0.9969$, which indicates a very strong, positive, linear relationship between voltage and current.

The Coefficient of Determination

The coefficient of determination, denoted by R^2 is a measure of how well a regression line, or curve, fits the provided data.[4] For linear regression (but not other types of regression) it is the square of the correlation coefficient, so $R^2 = r^2$. Its value is in the interval $[0, 1]$.

Example 4 For the electronic circuit example, $R^2 = 0.9937$, which indicates the regression model fits the data very well.

[4] A statistics class would teach you that R^2 is the proportion of the variation in the dependent variable which is explained by the model. Don't worry if that doesn't make any sense yet!

Making Predictions

There are two types of predictions that we can make using a regression model.

Interpolation means predicting values between the values in the data. If the model is a good fit for the data, then this can produce very reliable predictions.

Example 5 Estimate the current in the circuit when $V = 2.6\,\mathrm{V}$.

$$\begin{aligned} I &\approx 4.139(2.6) - 3.064 \\ &= 7.7\,\mathrm{mA} \end{aligned}$$

Example 6 Estimate the voltage that corresponds to a current of $I = 27.3\,\mathrm{mA}$.

$$\begin{aligned} 27.3 &\approx 4.139V - 3.064 \\ 4.139V &\approx 27.3 + 3.064 = 30.364 \\ V &\approx \frac{30.364}{4.139} = 7.3\,\mathrm{V} \end{aligned}$$

Extrapolation means predicting values outside the values in the data. You need to be careful when extrapolating, because it is very difficult to know how far the trend in the data continues outside of its range.

Example 7 Estimate the current in the circuit when $V = 0.3\,\mathrm{V}$.

$$\begin{aligned} I &\approx 4.139(0.3) - 3.064 \\ &= -1.8\,\mathrm{mA} \end{aligned}$$

Note that this prediction is unreliable.

For anyone who cares about the physics, the hypothetical circuit in this section is a silicon diode attached to a $250\,\Omega$ resistor in series. Not only is the negative current in the last example an unreliable result, it doesn't even make sense given the scenario.

2.5 Piecewise Linear Functions

A piecewise function is a function which is defined by multiple rules, each applying to different parts of the domain.

Example 1 Evaluate each of the following using the function f.

$$f(x) = \begin{cases} 2x & -2 \leq x \leq 3 \\ 4 & 3 < x < 6 \\ -x + 9 & x \geq 6 \end{cases}$$

$$f(1) = 2(1) \\ = 2$$

$$f(5) = 4$$

$$f(8) = -8 + 9 \\ = 1$$

$$f(6) = -6 + 9 \\ = 3$$

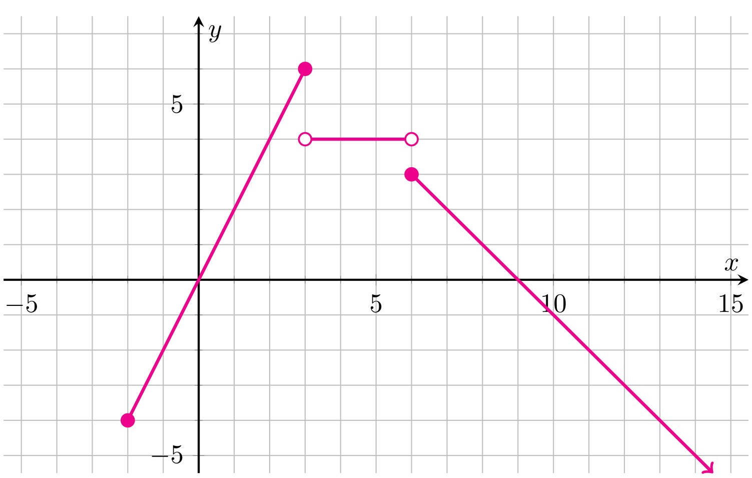
$$f(3) = 2(3) \\ = 6$$

$f(-3)$ is undefined

A piecewise function can be graphed by considering each rule separately, and plotting each on its own interval.

The domain of the entire piecewise function is the union of the domains of the separate rules. Similarly, the range is the union of the ranges produced by each rule.

Example 2 For function f above, plot its graph and find its domain and range.



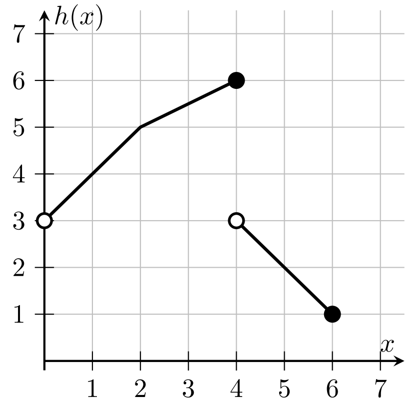
Domain:

$$[-2, 3] \cup (3, 6) \cup [6, \infty) \\ = [-2, \infty)$$

Range:

$$[-4, 6] \cup \{4\} \cup (-\infty, 3] \\ = (-\infty, 6]$$

Example 3 Define h as a piecewise function.



For $x \in (0, 2]$, $y = x + 3$

For $x \in (2, 4]$, $y = \frac{1}{2}(x - 2) + 5 = \frac{1}{2}x + 4$

For $x \in (4, 6]$, $y = -(x - 4) + 3 = -x + 7$

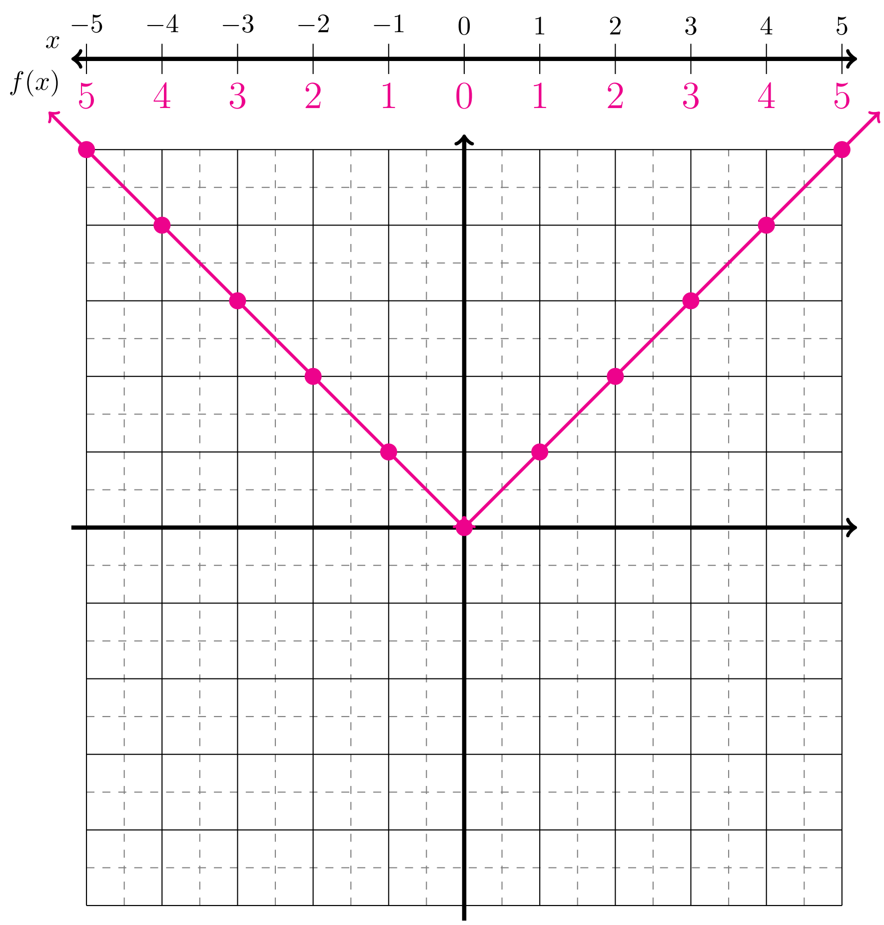
$$h(x) = \begin{cases} x + 3 & 0 < x \le 2 \\ \frac{1}{2}x + 4 & 2 < x \le 4 \\ -x + 7 & 4 < x \le 6 \end{cases}$$

The Absolute Value Parent Function

An important piecewise function is the absolute value function.

parent function $f(x) = \lvert x \rvert$
domain $\mathbb{R}$
range $[0, \infty)$
relation type many-to-one
x-intercept $(0, 0)$
y-intercept $(0, 0)$
vertex $(0, 0)$
slopes ± 1

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$



Absolute Value Functions

By applying transformations to the parent function, we get the general form of the absolute value function:

$$f(x) = A\,|x-h| + k$$

- Graph is upright or opens up if A is positive.
  Graph is inverted or opens down if A is negative.
- Graph has two linear intervals, whose slopes are $\pm A$.
- Graph has a vertex at (h, k).

A sketch of an absolute value function should include:

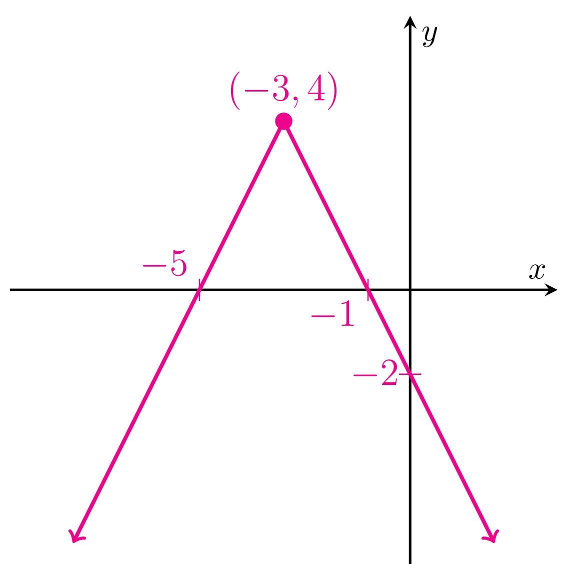
shape of curve	"V" shape with enough points to show slopes
vertex	(h, k), using translation of parent function to identify
x-intercepts	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$
endpoints	evaluate the function at the bounds of the domain

Example 4 Sketch $g(x) = -2\,|x+3| + 4$.

Orientation: Inverted

Slopes: $m = \pm 2$

Vertex: $(-3, 4)$



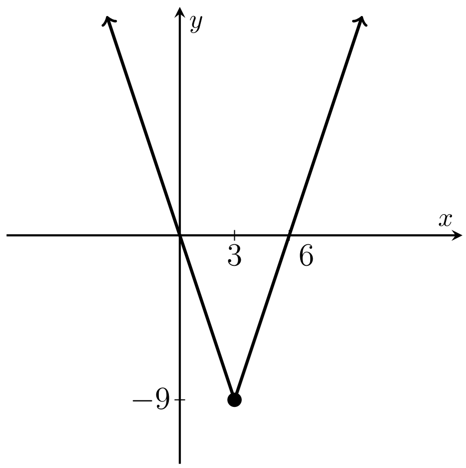
x-intercepts: $(-5, 0)$ and $(-1, 0)$

$$-2\,|x+3| + 4 = 0$$
$$-2\,|x+3| = -4$$
$$|x+3| = 2$$
$$x+3 = -2 \text{ or } x+3 = 2$$
$$x = -5 \text{ or } x = -1$$

y-intercept: $(0, -2)$
as $f(0) = -2\,|3| + 4 = -2$

endpoints: none, as domain is $\mathbb{R}$

Example 5 Find the function f represented by the following graph.



Orientation: Upright

Slopes: $m = \pm 3$

$\implies A = +3$

Vertex: $(3, -9)$

$\implies h = 3, k = -9$

$$f(x) = 3\,|x-3| - 9$$

Example 6 Find the range of $f : [2, 9) \to \mathbb{R}$, where $f(x) = \frac{1}{2}\,|x-4| + 3$.

The bounds of the range will occur at the endpoints or at the vertex.

At the vertex: $f(4) = 3$

Left endpoint: $f(2) = \frac{1}{2}\,|2-4| + 3 \quad = \frac{1}{2} \cdot 2 + 3 \quad = 4$

Right endpoint: $f(9)$ is undefined, $\frac{1}{2}\,|9-4| + 3 = \frac{1}{2} \cdot 5 + 3 \quad = \frac{11}{2}$

Range is $\left[3, \frac{11}{2}\right)$

Example 7 Find the transformations required to transform $f(x) = 2\,|x-2| + 1$ to $g(x) = -3\,|x+1| + 6$.

$$\begin{aligned} g(x) &= -3\,|x+1| + 6 \\ &= -\tfrac{3}{2} \cdot 2\,|(x+3) - 2| + 1 + 5 \\ &= -\tfrac{3}{2} f(x+3) + 5 \end{aligned}$$

- Reflect across the x-axis
- Stretch from the x-axis by a factor of $\frac{3}{2}$
- Shift 3 units left
- Shift 5 units up

Example 8 Express $f(x) = 5\,|x-4| + 7$ as a piecewise function.

When $x - 4 \geq 0$:

$$\begin{aligned} f(x) &= 5(x-4) + 7 \\ &= 5x - 20 + 7 \\ &= 5x - 13 \end{aligned}$$

When $x - 4 < 0$:

$$\begin{aligned} f(x) &= 5(-x+4) + 7 \\ &= -5x + 20 + 7 \\ &= -5x + 27 \end{aligned}$$

$$f(x) = \begin{cases} 5x - 13 & x \geq 4 \\ -5x + 27 & x < 4 \end{cases}$$

Chapter 3

Quadratic Functions and Equations

3.1 Quadratics in Vertex Form 42
3.2 Quadratics in Factored Form 46
3.3 Review of Distributing and Factoring 48
3.4 Special Quadratics 52
3.5 Factoring Quadratics in Standard Form 56
3.6 Completing the Square 60
3.7 The Quadratic Formula 63

3.1 Quadratics in Vertex Form

A quadratic expression is an expression which can be written in the form (with $a \neq 0$):

$$ax^2 + bx + c$$

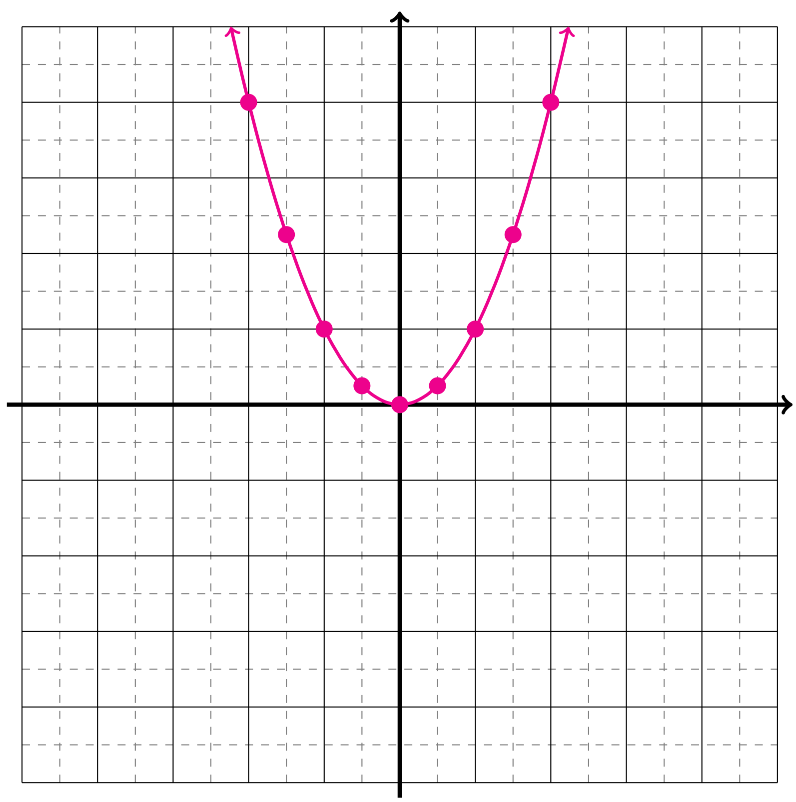
A quadratic function is a function consisting of a quadratic expression. The three forms of these functions we usually consider are

standard form	$f(x) = ax^2 + bx + c$
vertex form	$f(x) = A(x-h)^2 + k$
factored form	$f(x) = a(x+p)(x+q)$

The Quadratic Parent Function

parent function	$f(x) = x^2$
domain	$\mathbb{R}$
range	$[0, \infty)$
relation type	many-to-one
x-intercept	$(0,0)$
y-intercept	$(0,0)$
vertex	$(0,0)$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	25	16	9	4	1	0	1	4	9	16	25



Solving Quadratic Equations Using Square Roots

A quadratic equation is any equation which can be written with a quadratic expression on one side and zero on the other. Note that this might not be the original form of the equation.

If an equation is written in vertex form, it can be solved using square roots:

1. Rearrange the equation to isolate the quantity which is squared.
2. Eliminate the square with a square root. Consider both the positive and negative square roots.
3. Finish solving the equation by isolating x.

Example 1 Solve $2(x-4)^2 - 5 = 13$

$$2(x-4)^2 - 5 = 13$$
$$2(x-4)^2 = 18$$
$$(x-4)^2 = 9$$
$$x - 4 = \pm\sqrt{9} = \pm 3$$
$$x = 4 \pm 3$$
$$x = 1 \text{ or } x = 7$$

Example 2 Solve $-3(x+5)^2 + 7 = 7$

$$-3(x+5)^2 + 7 = 7$$
$$-3(x+5)^2 = 0$$
$$(x+5)^2 = 0$$
$$x + 5 = 0$$
$$x = -5$$

Example 3 Solve $(x+2)^2 - 7 = 0$

$$(x+2)^2 - 7 = 0$$
$$(x+2)^2 = 7$$
$$(x+2)^2 = \pm\sqrt{7}$$
$$x = -2 \pm \sqrt{7}$$

Example 4 Solve $2(x-6)^2 + 9 = 1$

$$2(x-6)^2 + 9 = 1$$
$$2(x-6)^2 = -8$$
$$(x-6)^2 = -4$$
$$\implies \text{no real solution}$$

Note that quadratic equations may have zero, one, or two real[1] solutions.

[1] In an upcoming lesson, you will see that it is possible to get solutions that are not real numbers! For now, we're only considering the real numbers.

Graphing Quadratic Functions Using Vertex Form

By applying transformations to the quadratic parent function, we get the vertex form of a quadratic function:

$$f(x) = A(x-h)^2 + k$$

- Graph is upright or opens up if A is positive.
  Graph is inverted or opens down if A is negative.
- $|A|$ corresponds to a stretch or compression from the x-axis.
- Graph has a vertex at (h, k).

A sketch of a quadratic function should include:

shape of curve	parabola with enough points to show stretch/compression
vertex	(h, k), using translation of parent function to identify
x-intercepts	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$
endpoints	evaluate the function at the bounds of the domain

Example 5 Sketch $f(x) = (x-3)^2 - 4$.

Orientation: Upright

Vertex: $(3, -4)$

x-intercepts: $(-5, 0)$ and $(-1, 0)$

$(x-3)^2 - 4 = 0$

$(x-3)^2 = 4$

$x - 3 = \pm\sqrt{4} = \pm 2$

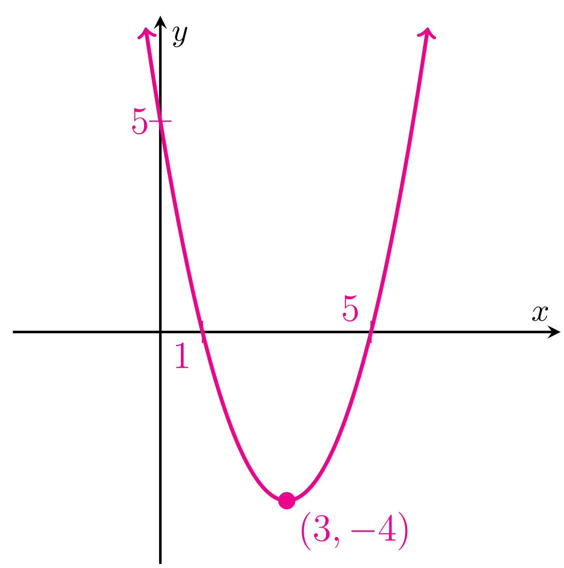
$x = 3 \pm 2$

$x = 1$ or $x = 5$

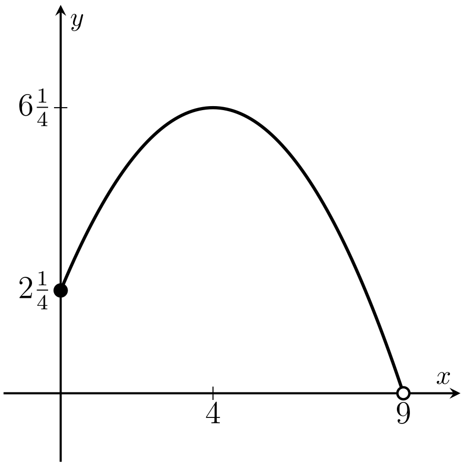
y-intercept: $(0, 5)$

as $f(0) = (-3)^2 - 4 = 5$

endpoints: none, as domain is $\mathbb{R}$



Example 6 Find the function g represented by the following graph.



Vertex: $(4, \frac{25}{4}) \implies h = 4, k = \frac{25}{4}$

$g(x) = A(x-4)^2 + \frac{25}{4}$

y-intercept: $(0, \frac{9}{4})$

$g(0) = 16A + \frac{25}{4} = \frac{9}{4}$

$16A = -4 \implies A = -\frac{1}{4}$

Domain: $[0, 9)$

$g : [0,9) \to \mathbb{R}$, where
$g(x) = -\frac{1}{4}(x-4)^2 + \frac{25}{4}$

Example 7 Find the range of $h : [-3, 1] \to \mathbb{R}$, where $h(x) = -2(x+2)^2 + 7$.

The bounds of the range will occur at the endpoints or at the vertex.

At the vertex: $f(-2) = 7$

Left endpoint: $f(-3) = -2(-3+2)^2 + 7 \quad = -2 \cdot 1 + 7 \quad = 5$

Right endpoint: $f(1) = -2(1+2)^2 + 7 \quad = -2 \cdot 9 + 7 \quad = -11$

Range is $[-11, 7]$

Zeros, Roots, Solutions and x-Intercepts

These terms are related, but have subtly different meanings.

The roots of an **expression** are the values which cause the expression to equal zero.

The solutions of an **equation** are the values which cause the equation to be true.

The zeros of a **function** are the input values which cause the output value to be zero.

The x-intercepts of a **graph** are the points where the curve crosses the x-axis.

Example 8 (Working in Example 5.)

The solutions of $(x-3)^2 - 4 = 0$ are 1 and 5.

The zeros of $f(x) = (x-3)^2 - 4$ are 1 and 5.

The roots of $(x-3)^2 - 4$ are 1 and 5.

The x-intercepts of the graph of $y = (x-3)^2 - 4$ are $(1, 0)$ and $(5, 0)$.

3.2 Quadratics in Factored Form

The Zero Product Property

If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.

Equivalently, if the product of a set of factors is zero, then at least one of the factors is zero.

Quadratic Equations in Factored Form

Example 1 Solve $3x(x-5) = 0$

$$3x(x-5) = 0$$
$$3x = 0 \text{ or } x - 5 = 0$$
$$x = 0 \text{ or } x = 5$$

Example 2 Solve $(x-4)(x+7) = 0$

$$(x-4)(x+7) = 0$$
$$x - 4 = 0 \text{ or } x + 7 = 0$$
$$x = 4 \text{ or } x = -7$$

Example 3 Solve $(5x-2)(7x+4) = 0$

$$(5x-2)(7x+4) = 0$$
$$5x - 2 = 0 \text{ or } 7x + 4 = 0$$
$$5x = 2 \text{ or } 7x = -4$$
$$x = \tfrac{2}{5} \text{ or } x = -\tfrac{4}{7}$$

Example 4 Solve $(3x-8)^2 = 0$

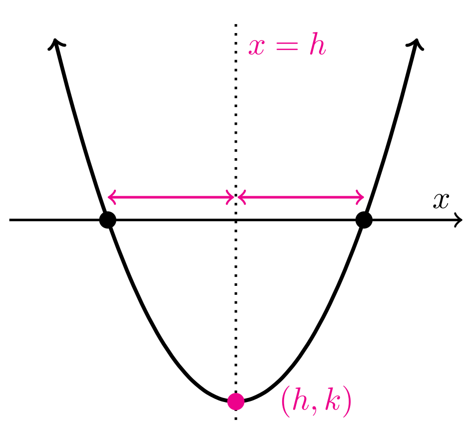
$$(3x-8)(3x-8) = 0$$
$$3x - 8 = 0$$
$$3x = 8$$
$$x = \tfrac{8}{3}$$

Graphing Quadratic Functions in Factored Form

We can use the zero product property as above to find the x-intercepts of the graph.

To find the vertex, we can use the symmetry of the parabola. The axis of symmetry passes through the vertex, as well as exactly halfway between the x-intercepts.

h is the average of the zeros of the function, and k is the value of the function evaluated at h.



Example 5 Sketch a graph of $f(x) = (x-2)(x-10)$.

x-intercepts: $(2,0)$ and $(10,0)$

$f(x) = 0 \implies x = 2$ or $x = 10$

y-intercept: $(0,20)$

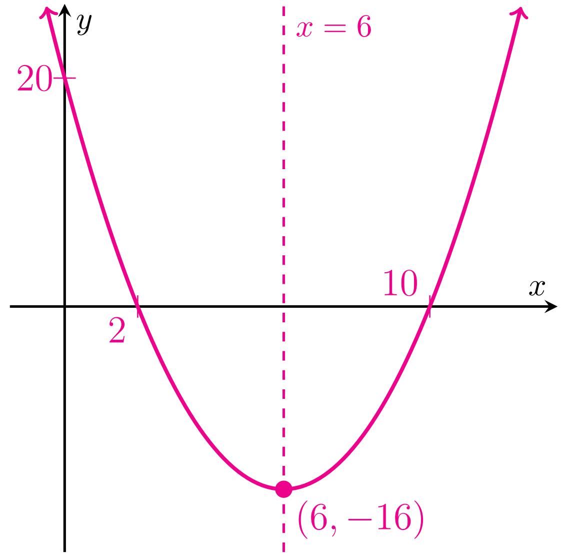
$f(0) = (-2)(-10) = 20$

vertex: $(6,-16)$

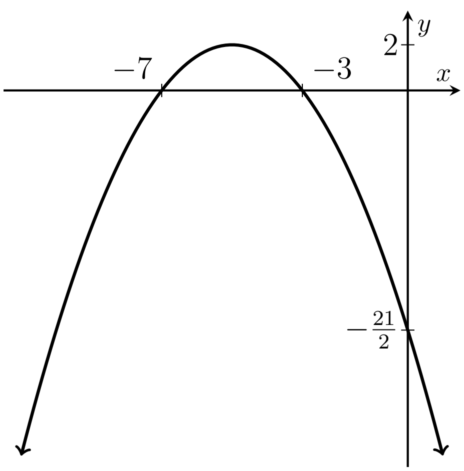
$$h = \frac{2+10}{2} = 6$$

$k = f(h) = (6-2)(6-10) = -16$

endpoints: none, as domain is $\mathbb{R}$



Example 6 Find the function g represented by the following graph.



x-intercepts: $(-7,0)$ and $(-3,0)$

$g(x) = a(x+7)(x+3)$

y-intercept: $\left(0, -\frac{21}{2}\right)$

$g(0) = a(7)(3) = 21a = -\frac{21}{2}$

$a = -\frac{1}{2}$

$$g(x) = -\frac{1}{2}(x+7)(x+3)$$

Example 7 Write $f(x) = (1-x)(x+6)$ in vertex form.

Zeros of f: $(1-x)(x+6) = 0$

$\implies x = 1$ or $x = -6$

$$h = \frac{1+(-6)}{2} = -\frac{5}{2}$$

$k = f(h) = \left(1 + \frac{5}{2}\right)\left(-\frac{5}{2} + 6\right)$

$= \frac{7}{2} \cdot \frac{7}{2} = \frac{49}{4}$

$f(x) = A\left(x + \frac{5}{2}\right)^2 + \frac{49}{4}$

$f(1) = 0 \implies A\left(1 + \frac{5}{2}\right)^2 + \frac{49}{4} = 0$

$\frac{49}{4}A + \frac{49}{4} = 0$

$\frac{49}{4}A = -\frac{49}{4}$

$A = -1$

$$f(x) = -\left(x + \frac{5}{2}\right)^2 + \frac{49}{4}$$

3.3 Review of Distributing and Factoring

The distributive property is one of the most important rules in algebra. Many of our results going forward are derived from it.

The Distributive Property

$$a(b+c) = ab + ac$$

Example 1 Verify $8(7+5) = 8 \cdot 7 + 8 \cdot 5$

$$\begin{aligned} 8(7+5) &= 8 \cdot 12 \\ &= 96 \\ 8 \cdot 7 + 8 \cdot 5 &= 56 + 40 \\ &= 96 \end{aligned}$$

Example 2 Verify $3(20-6) = 3 \cdot 20 - 3 \cdot 6$

$$\begin{aligned} 3(20-6) &= 3 \cdot 14 \\ &= 42 \\ 3 \cdot 20 - 3 \cdot 6 &= 60 - 18 \\ &= 42 \end{aligned}$$

The process of changing $a(b+c)$ to $ab+ac$ is called distributing.

The reverse process is called factoring.

The box method can be used to visualize the distributive property.

	b	$+c$
a	ab	$+ac$

Distributing

To distribute algebraically, multiply each term inside the parentheses by the factor outside the parentheses.

Example 3 Distribute $3x(2x-4)$

	$2x$	-4
$3x$	$6x^2$	$-12x$

$$3x(2x-4) = 6x^2 - 12x$$

Example 4 Distribute $-4y(7y^2+5)$

	$7y^2$	$+0y$	$+5$
$-4y$	$-28y^3$	$-0y^2$	$-20y$

$$-4y(7y^2+5) = -28y^3 - 20y$$

Example 5 Distribute $3x^2(x^4 - 2x^3 + 5x - 1)$

	x^4	$-2x^3$	$+0x^2$	$+5x$	-1
$3x^2$	$3x^6$	$-6x^5$	$+0x^4$	$+15x^3$	$-3x^2$

$$3x^2(x^4 - 2x^3 + 5x - 1) = 3x^6 - 6x^5 + 15x^3 - 3x^2$$

If there are two sets of parentheses, we need to distribute over both. Every term in the first set of parentheses is multiplied by every term in the second set of parentheses. After distributing, make sure you combine like terms.

Example 6 Distribute $(x + 4)(x - 7)$

	x	$+4$
x	x^2	$+4x$
-7	$-7x$	-28

$$(x + 4)(x - 7) = x^2 - 3x - 28$$

Example 7 Distribute $(2x + 3)(x + 6)$

	$2x$	$+3$
x	$2x^2$	$+3x$
$+6$	$+12x$	$+18$

$$(2x + 3)(x + 6) = 2x^2 + 15x + 18$$

Example 8 Distribute $(3x - 5)(x^3 + 2x^2 - 7)$

	x^3	$2x^2$	$+0x$	-7
$3x$	$3x^4$	$+6x^3$	$+0x^2$	$-21x$
-5	$-5x^3$	$-10x^2$	$-0x$	$+35$

$$(3x - 5)(x^3 + 2x^2 - 7) = 3x^4 + x^3 - 10x^2 - 21x + 35$$

Factoring Using the Greatest Common Factor

If all the terms in an expression have a factor which is the same, that factor is called a common factor.

The greatest common factor, or GCF, is the largest possible common factor for the expression.

To factor, we can divide every term by the GCF, and write the result in parentheses, with the GCF written in front. As the expression has been both divided and multiplied by the GCF, the result is equivalent.

This method of factoring is the simplest and should be attempted first. If this is done correctly, there will be no common factors remaining.

Example 9 Factor $9m^3 - 12m^2$

	$3m$	-4
$3m^2$	$9m^3$	$-12m^2$

$$9m^3 - 12m^2 = 3m^2(3m - 4)$$

Example 10 Factor $12a^3b + 24a^2b^5 - 42a^4b^4$

	$2a$	$+4b^4$	$-7a^2b^3$
$6a^2b$	$12a^3b$	$+24a^2b^5$	$-42a^4b^4$

$$12a^3b + 24a^2b^5 - 42a^4b^4 = 6a^2b(2a + 4b^4 - 7a^2b^3)$$

Quadratics with Common Factors

We've already seen that factored form can be convenient for finding the zeros of a function. In certain circumstances, factoring the GCF can change a quadratic expression/function in standard form into factored form.

Example 11 Solve $15x^2 + 10x = 0$

$$15x^2 + 10x = 0$$
$$5x(3x+2) = 0$$
$$5x = 0 \text{ or } 3x + 2 = 0$$
$$x = 0 \text{ or } 3x = -2$$
$$x = 0 \text{ or } x = -\tfrac{2}{3}$$

Example 12 Solve $2x^2 = 8x$

$$2x^2 = 8x$$
$$2x^2 - 8x = 0$$
$$2x(x-4) = 0$$
$$2x = 0 \text{ or } x - 4 = 0$$
$$x = 0 \text{ or } x = 4$$

Example 13 Sketch a graph of $f(x) = -3x^2 - 15x$.

factor: $f(x) = -3x(x+5)$

x-intercepts: $(0,0)$ and $(-5,0)$

$f(x) = 0 \implies x = 0$ or $x = -5$

y-intercept: $(0,0)$

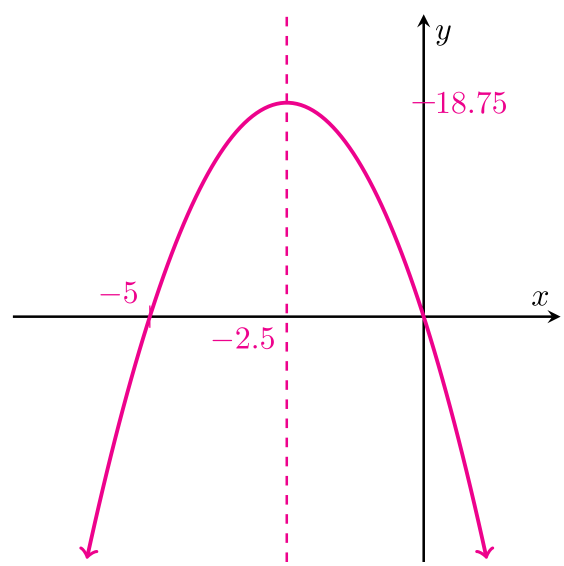
$f(0) = -3(0)^2 - 15(0) = 0$

vertex: $(-2.5, 18.75)$

$$h = \frac{0 + (-5)}{2} = -2.5$$

$$\begin{aligned} k &= f(h) \\ &= -3(-2.5)^2 - 15(-2.5) \\ &= 18.75 \end{aligned}$$

endpoints: none, as domain is $\mathbb{R}$



3.4 Special Quadratics

In the previous section, we factored select quadratics in standard form using the greatest common factor. The following rules will allow us to factor other special cases.

Theorem: Perfect Squares

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Proof

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

	a	$+b$
a	a^2	$+ab$
$+b$	$+ab$	$+b^2$

Replace b with $-b$ to obtain the second result. ■

Theorem: Differences of Squares

$$a^2 - b^2 = (a+b)(a-b)$$

Proof

$$\begin{aligned}(a+b)(a-b) &= a(a-b) + b(a-b)\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

■

	a	$+b$
a	a^2	$+ab$
$-b$	$-ab$	$-b^2$

These rules can be used for distributing:

Example 1 Distribute $(x+10)^2$

Using $a = x$ and $b = 10$,

$$(x+10)^2 = x^2 + 20x + 100$$

Example 2 Distribute $(2x+7)(2x-7)$

Using $a = 2x$ and $b = 7$,

$$(2x+7)(2x-7) = 4x^2 - 49$$

The rules can also be used for factoring:

Example 3 Factor $x^2 - 81$

Using $a = x$ and $b = 9$,

$$x^2 - 81 = (x+9)(x-9)$$

Example 4 Factor $25x^2 - 30x + 9$

Using $a = 5x$ and $b = 3$,

$$25x^2 - 30x + 9 = (5x-3)^2$$

It is always a good idea to attempt to factor using the GCF before factoring with any other method, including special quadratics:

Example 5 Factor $5x^2 + 20x + 20$

$$\begin{aligned} 5x^2 + 20x + 20 &= 5(x^2 + 4x + 4) \\ &= 5(x+2)^2 \end{aligned}$$

Example 6 Factor $63x^2 - 175$

$$\begin{aligned} 63x^2 - 175 &= 7(9x^2 - 25) \\ &= 7(3x+5)(3x-5) \end{aligned}$$

As with all quadratic equations, equations in these forms can be solved using the zero product property if they are factored:

Example 7 Solve $4x^2 + 196 = 56x$

$$\begin{aligned} 4x^2 - 56x + 196 &= 0 \\ 4(x^2 - 14x + 49) &= 0 \\ x^2 - 14x + 49 &= 0 \\ (x-7)^2 &= 0 \\ x &= 7 \end{aligned}$$

Example 8 Solve $12x^2 - 75 = 0$

$$\begin{aligned} 3(4x^2 - 25) &= 0 \\ 4x^2 - 25 &= 0 \\ (2x+5)(2x-5) &= 0 \\ 2x &= \pm 5 \\ x &= \pm \tfrac{5}{2} \end{aligned}$$

Perfect Squares and Differences of Squares as Functions

Note that the perfect squares and differences of squares rules are useful for converting these types of quadratic functions between their three forms:

	perfect square	difference of squares
standard form	$f(x) = x^2 + 2mx + m^2$	$g(x) = x^2 - n^2$
vertex form	$f(x) = (x+m)^2$	$g(x) = x^2 - n^2$
factored form	$f(x) = (x+m)^2$	$g(x) = (x+n)(x-n)$

Example 9 Sketch a graph of $f(x) = -2x^2 + 12x - 18$.

factor: $f(x) = -2(x^2 - 6x + 9) = -2(x-3)^2$

x-intercepts: $(3, 0)$

$f(x) = 0 \implies x = 3$

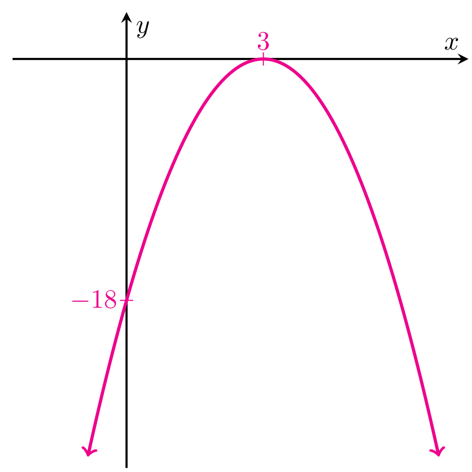
y-intercept: $(0, -18)$

$f(0) = -18$

vertex: $(3, 0)$

$h = 3, \quad k = 0$

endpoints: none, as domain is $\mathbb{R}$



Example 10 Sketch a graph of $f(x) = 3x^2 - 12$.

factor: $f(x) = 3(x^2 - 4) = 3(x+2)(x-2)$

x-intercepts: $(-2, 0)$ and $(2, 0)$

$f(x) = 0 \implies x = \pm 2$

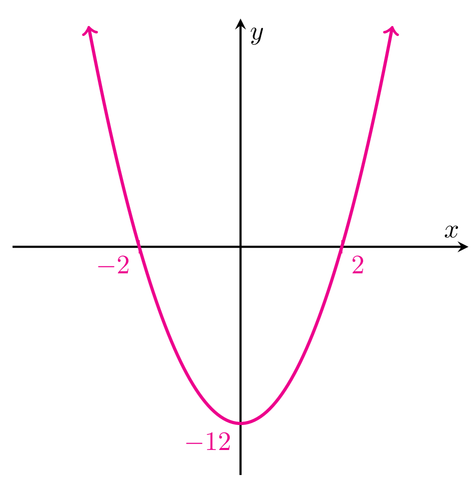
y-intercept: $(0, -12)$

$f(0) = -12$

vertex: $(0, -12)$

$h = 0, \quad k = -12$

endpoints: none, as domain is $\mathbb{R}$



Example 11 Write $g(x) = (x-5)^2 - 9$ in factored form.

This is a difference of squares with $a = x-5$ and $b = 3$.

$$\begin{aligned} g(x) &= (x-5)^2 - 3^2 \\ &= (x-5+3)(x-5-3) \\ &= (x-2)(x-8) \end{aligned}$$

Example 12 Write $h(x) = (x+7)^2 - 12$ in factored form.

This is a difference of squares with $a = x+7$ and $b = \sqrt{12} = 2\sqrt{3}$.

$$\begin{aligned} h(x) &= (x+7)^2 - \left(2\sqrt{3}\right)^2 \\ &= \left(x+7+2\sqrt{3}\right)\left(x+7-2\sqrt{3}\right) \end{aligned}$$

Further Factoring Examples

While perfect squares and differences of squares are examples of quadratic expressions, they can also be used to factor certain other polynomials[2].

Example 13 Factor $8x^4 - 18x^2$

$$\begin{aligned} 8x^4 - 18x^2 &= 2x^2(4x^2-9) \\ &= 2x^2(2x+3)(2x-3) \end{aligned}$$

Example 14 Solve $5x^3 + 60x^2 + 180x = 0$

$$\begin{aligned} 5x^3 + 60x^2 + 180x &= 0 \\ 5x(x^2+12x+36) &= 0 \\ 5x(x+6)^2 &= 0 \\ x = 0 \text{ or } x &= -6 \end{aligned}$$

Example 15 Factor $x^4 - 18x^2 + 81$

Let $u = x^2$

$$\begin{aligned} x^4 - 18x^2 + 81 &= u^2 - 18u + 81 \\ &= (u-9)^2 \\ &= (x^2-9)^2 \\ &= [(x+3)(x-3)]^2 \\ &= (x+3)^2(x-3)^2 \end{aligned}$$

[2] We'll discuss polynomials in detail in a later chapter.

3.5 Factoring Quadratics in Standard Form

Recall that the standard form of a quadratic expression is

$$ax^2 + bx + c$$

Factoring Monic Quadratics

A quadratic expression is called monic if $a = 1$.

Theorem

If a monic quadratic expression $x^2 + bx + c$ has values p and q such that

$$b = p + q \qquad \text{and} \qquad c = p \cdot q$$

then

$$x^2 + bx + c = (x + p)(x + q)$$

Proof

$$\begin{aligned} x^2 + bx + c &= x^2 + (p+q)x + pq \\ &= x^2 + px + qx + pq \\ &= x(x+p) + q(x+p) \\ &= (x+p)(x+q) \qquad \blacksquare \end{aligned}$$

	x	$+p$
x	x^2	$+px$
$+q$	$+qx$	$+pq$

Example 1 Factor $x^2 + 7x + 12$

	x	$+3$
x	x^2	$+3x$
$+4$	$+4x$	$+12$

$$x^2 + 7x + 12 = (x+3)(x+4)$$

Example 2 Factor $x^2 - 3x - 40$

	x	-8
x	x^2	$-8x$
$+5$	$+5x$	-40

$$x^2 - 3x - 40 = (x-8)(x+5)$$

Factoring Non-monic Quadratics

Often, a non-monic quadratic can be factored as if it were monic by first factoring using the GCF.

Example 3 Factor $6x^2 - 30x + 36$

$$\begin{aligned} &6x^2 - 30x + 36 \\ &\quad = 6(x^2 - 5x + 6) \\ &\quad = 6(x-2)(x-3) \end{aligned}$$

Example 4 Solve $-4x^2 + 36x + 88$

$$\begin{aligned} &-4x^2 + 36x + 88 \\ &\quad = -4(x^2 - 9x - 22) \\ &\quad = -4(x-11)(x+2) \end{aligned}$$

If this is not an option, then the following theorem can be used to help factor using the box method.

Theorem

In a 2×2 box using the box method, the products of the values along each diagonal are the same.

Proof

Consider the general expression $(a+b)(c+d)$, which is distributed using the box method.

Along the first diagonal: $ac \cdot bd = abcd$

Along the second diagonal: $bc \cdot ad = abcd$ ■

	a	$+b$
c	ac	$+bc$
$+d$	$+ad$	$+bd$

Example 5 Factor $5x^2 + 28x - 12$

The first diagonal contains $5x^2$ and -12.

The second diagonal has sum $28x$ and product $-60x^2$.

$\implies$ second diagonal is $-2x$ and $30x$.

Finding common factors for each row and column gives

$$5x^2 + 28x - 12 = (5x-2)(x+6)$$

	$5x$	-2
x	$5x^2$	$-2x$
$+6$	$+30x$	-12

Example 6 Factor $12x^2 - 24x - 15$

	$2x$	-5
$2x$	$4x^2$	$-10x$
$+1$	$+2x$	-5

$$\begin{aligned} &12x^2 - 24x - 15 \\ &= 3(4x^2 - 8x - 5) \\ &= 3(2x-5)(2x+1) \end{aligned}$$

Example 7 Factor $-12x^2 + 58x - 18$

	$3x$	-1
$2x$	$6x^2$	$-2x$
-9	$-27x$	$+9$

$$\begin{aligned} &-12x^2 + 58x - 18 \\ &= -2(6x^2 - 29x + 9) \\ &= -2(3x-1)(2x-9) \end{aligned}$$

Solving Equations by Factoring

Recall that a __solution__ to an equation is a value which causes it to be __true__. For quadratic equations, __factoring__ allows us to use the __zero product property__ to find the solutions.

Example 8 Solve $x^2 + 15x + 36 = 0$

$$\begin{aligned} x^2 + 15x + 36 &= 0 \\ (x+3)(x+12) &= 0 \\ x+3 = 0 \text{ or } x+12 &= 0 \\ x = -3 \text{ or } x &= -12 \end{aligned}$$

Example 9 Solve $x^2 + 5 = 8x + 14$

$$\begin{aligned} x^2 + 5 &= 8x + 14 \\ x^2 - 8x - 9 &= 0 \\ (x-9)(x+1) &= 0 \\ x - 9 = 0 \text{ or } x + 1 &= 0 \\ x = 9 \text{ or } x &= -1 \end{aligned}$$

Example 10 Solve $4x^2 + 25x - 21 = 0$

$$\begin{aligned} 4x^2 + 25x - 21 &= 0 \\ (4x-3)(x+7) &= 0 \\ 4x - 3 = 0 \text{ or } x + 7 &= 0 \\ x = \tfrac{3}{4} \text{ or } x &= -7 \end{aligned}$$

Example 11 Solve $20x^2 - 56x - 12 = 0$

$$\begin{aligned} 20x^2 - 56x - 12 &= 0 \\ \cancel{4}(5x^2 - 14x - 3) &= 0 \\ (5x+1)(x-3) &= 0 \\ 5x + 1 = 0 \text{ or } x - 3 &= 0 \\ x = -\tfrac{1}{5} \text{ or } x &= 3 \end{aligned}$$

Graphing Using Factoring

We've already graphed quadratic functions in factored form. Using the same methods, we can graph quadratic functions in standard form if they can be factored.

Example 12 Sketch a graph of $f(x) = x^2 + x - 2$.

factor: $f(x) = (x+2)(x-1)$

x-intercepts: $(-2, 0)$ and $(1, 0)$

$f(x) = 0 \implies x = -2$ or $x = 1$

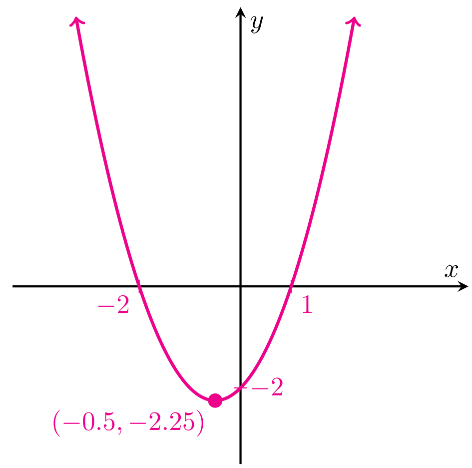
y-intercept: $(0, -2)$

vertex: $(-0.5, -2.25)$

$h = \frac{(-2)+1}{2} = -0.5$

$k = f(h) = (-0.5)^2 + (-0.5) - 2 = -2.25$

endpoints: none, as domain is $\mathbb{R}$



Example 13 Sketch a graph of $g(x) = -2x^2 + 9x - 9$.

factor: $g(x) = (-2x+3)(x-3)$

x-intercepts: $(1.5, 0)$ and $(3, 0)$

$g(x) = 0 \implies x = 1.5$ or $x = 3$

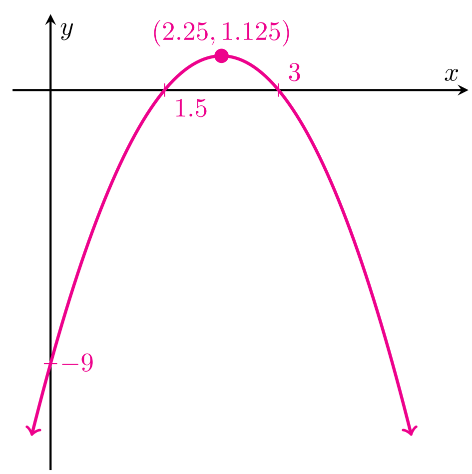
y-intercept: $(0, -9)$

vertex: $(2.75, 1.125)$

$h = \frac{2.5+3}{2} = 2.75$

$$k = g(h) = -2(2.75)^2 + 9(2.75) - 9$$
$$= 1.125$$

endpoints: none, as domain is $\mathbb{R}$

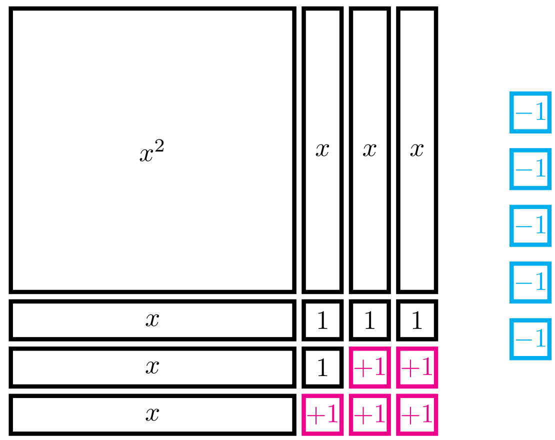


3.6 Completing the Square

While many quadratic expressions can be factored directly using the methods in the previous sections, most cannot. Instead, we use can a technique called completing the square.

The goal is to rewrite the expression so that it contains a perfect square, which is then factored. The result is an expression in vertex form. This makes it possible to solve the related equation, or graph the related function.

The diagram to the right shows that $x^2 + 6x + 4$ is not a perfect square, but its square can be completed by adding and subtracting 5.



Example 1 Solve $x^2 + 6x + 4 = 0$ by completing the square.

Step 1: Identify the constant which completes the square.	$x^2 + 6x \underbrace{+4}_{\text{want to be } +9} = 0$
Step 2: Add and subtract to complete the perfect square.	$\underbrace{x^2 + 6x + 9}_{\text{perfect square}} - 5 = 0$
Step 3: Factor the perfect square to get vertex form.	$(x+3)^2 - 5 = 0$
Step 4: Solve using the square root method.	$(x+3)^2 = 5$ $x + 3 = \pm\sqrt{5}$ $x = -3 \pm \sqrt{5}$

Example 2 Solve $x^2 - 10x + 7 = 0$

$$x^2 - 10x \underbrace{+7}_{\text{want to be } +25} = 0$$

$$x^2 - 10x + 25 - 18 = 0$$

$$(x-5)^2 - 18 = 0$$

$$(x-5)^2 = 18$$

$$x - 5 = \pm 3\sqrt{2}$$

$$x = 5 \pm 3\sqrt{2}$$

Example 3 Solve $x^2 + 2x - 5 = 0$

$$x^2 + 2x \underbrace{-5}_{\text{want to be } +1} = 0$$

$$x^2 + 2x + 1 - 6 = 0$$

$$(x+1)^2 - 6 = 0$$

$$(x+1)^2 = 6$$

$$x + 1 = \pm\sqrt{6}$$

$$x = -1 \pm \sqrt{6}$$

Example 4 Solve $x^2 + 3x + 1 = 0$

$$x^2 + 3x \underbrace{+1}_{\text{want to be } +\frac{9}{4}} = 0$$

$$x^2 + 3x + \tfrac{9}{4} - \tfrac{5}{4} = 0$$

$$\left(x + \tfrac{3}{2}\right)^2 - \tfrac{5}{4} = 0$$

$$\left(x + \tfrac{3}{2}\right)^2 = \tfrac{5}{4}$$

$$x + \tfrac{3}{2} = \pm\tfrac{\sqrt{5}}{2}$$

$$x = -\tfrac{3}{2} \pm \tfrac{\sqrt{5}}{2}$$

Example 5 Solve $4x^2 + 20x + 18 = 0$

$$4x^2 + 20x \underbrace{+18}_{\text{want to be } +25} = 0$$

$$4x^2 + 20x + 25 - 7 = 0$$

$$(2x+5)^2 - 7 = 0$$

$$(2x+5)^2 = 7$$

$$2x + 5 = \pm\sqrt{7}$$

$$2x = -5 \pm \sqrt{7}$$

$$x = -\tfrac{5}{2} \pm \tfrac{\sqrt{7}}{2}$$

Example 6 Write $f(x) = x^2 - 8x + 13$ in vertex form.

$$\begin{aligned} f(x) &= x^2 - 8x + 13 \\ &= x^2 - 8x + 16 - 3 \\ &= (x-4)^2 - 3 \end{aligned}$$

Example 7 Write $g(x) = -2x^2 - 20x - 59$ in vertex form.

$$\begin{aligned} g(x) &= -2x^2 - 20x - 59 \\ &= -2\left(x^2 + 10x + \tfrac{59}{2}\right) \\ &= -2\left(x^2 + 10x + 25 + \tfrac{9}{2}\right) \\ &= -2\left[(x+5)^2 + \tfrac{9}{2}\right] \\ &= -2(x+5)^2 - 9 \end{aligned}$$

Example 8 Sketch a graph of $f(x) = x^2 - 6x + 1$.

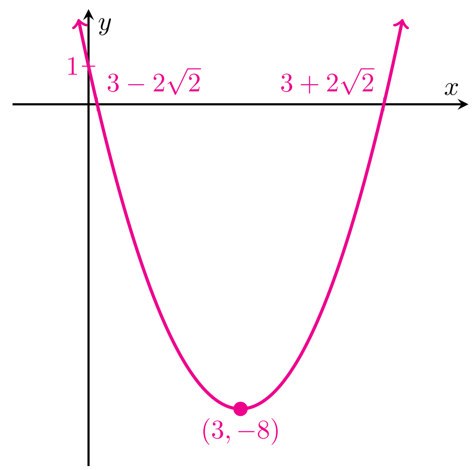
x-intercepts: $(3 - 2\sqrt{2}, 0)$ and $(3 + 2\sqrt{2}, 0)$

$$\begin{aligned} f(x) &= x^2 - 6x + 9 - 8 \\ &= (x-3)^2 - 8 \\ &= 0 \end{aligned}$$

$$(x-3)^2 = 8$$

$$x - 3 = \pm 2\sqrt{2}$$

$$x = 3 \pm 2\sqrt{2}$$



y-intercept: $(0, 1)$

vertex: $(3, -8)$

endpoints: none, as domain is $\mathbb{R}$

3.7 The Quadratic Formula

An alternative method to completing the square is using a formula to directly find the solutions to a quadratic equation.

Theorem: The Quadratic Formula

A quadratic equation in standard form, $ax^2 + bx + c = 0$, can be solved directly using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{divide both sides by } a \quad (1) \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 && \text{complete the square} \quad (2) \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} &= 0 && \text{factor and simplify} \quad (3) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && \text{isolate squared expression} \quad (4) \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{take the square root} \quad (5) \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{finish solving for } x \quad (6)
\end{aligned}$$

■

The quantity $b^2 - 4ac$ is known as the discriminant, denoted by Δ, the upper case Greek letter delta. We can use it to state a simplified version of the quadratic formula.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \qquad \text{where} \qquad \Delta = b^2 - 4ac$$

Example 1 Solve $2x^2 + x - 28 = 0$

$$a = 2, \quad b = 1, \quad c = -28$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(1) \pm \sqrt{(1)^2 - 4(2)(-28)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{225}}{4}$$

$$= \frac{-1 \pm 15}{4}$$

$$x = \frac{-1 - 15}{4} \text{ or } x = \frac{-1 + 15}{4}$$

$$x = -4 \text{ or } x = \tfrac{7}{2}$$

Example 2 Solve $3x^2 = 2x + 2$

$$3x^2 - 2x - 2 = 0$$

$$a = 3, \quad b = -2, \quad c = -2$$

$$\Delta = b^2 - 4ac$$

$$= (-2)^2 - 4(3)(-2)$$

$$= 28$$

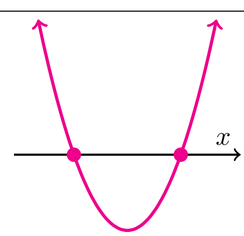
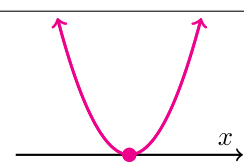
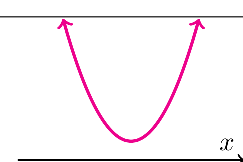
$$x = \frac{-(-2) \pm \sqrt{28}}{2(3)}$$

$$= \frac{2 \pm 2\sqrt{7}}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{7}}{3}$$

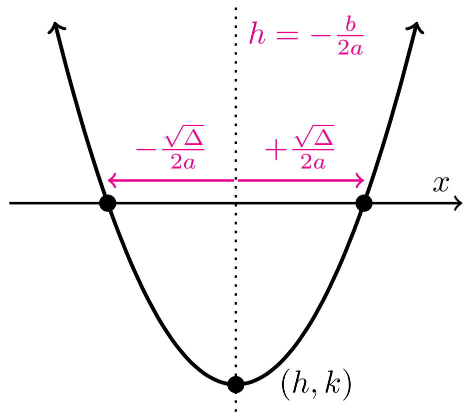
Counting Real Solutions

The sign of the discriminant is particularly useful for finding the number of real solutions to a quadratic equation. This also corresponds to the number of x-intercepts in the graph of a quadratic function.

	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
solutions	$\frac{-b \pm \sqrt{+ve}}{2a}$	$\frac{-b}{2a}$	$\frac{-b \pm \sqrt{-ve}}{2a}$
number of real solutions	two	one	zero
x-intercepts			

Graphing Quadratic Functions in Standard Form

Recall that the x-coordinate of the vertex, h, is the average of the zeros of the function.



Since the zeros of the function are given by the quadratic formula, we get that their average is given by

$$h = -\frac{b}{2a}$$

This formula holds even if there are not two real zeros.

This gives us the final tools we need for graphing quadratic functions in standard form.

shape of curve	parabola with enough points to show stretch/compression
vertex	(h, k), using $h = -\frac{b}{2a}$ and $k = f(h)$
x-intercepts	$y = 0$, find x by solving $f(x) = 0$ using factoring, completing the square, or quadratic formula
y-intercept	$(0, c)$
endpoints	evaluate the function at the bounds of the domain

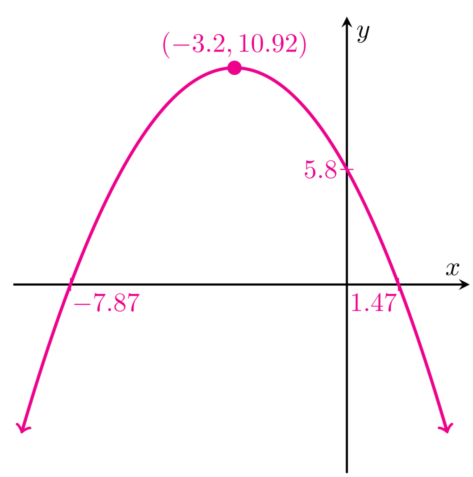
Example 3 Sketch a graph of $f(x) = -0.5x^2 - 3.2x + 5.8$, with x-intercepts to 2 decimal places.

$a = -0.5, \quad b = -3.2, \quad c = 5.8$

x-intercepts: $(-7.87, 0)$ and $(1.47, 0)$

$$x = \frac{-(-3.2) \pm \sqrt{(-3.2)^2 - 4(-0.5)(5.8)}}{2(-0.5)}$$

$$= -7.87 \text{ or } 1.47$$



y-intercept: $(0, 5.8)$

vertex: $(-3.2, 10.92)$

$h = -\frac{-3.2}{2(-0.5)} = -3.2$

$k = f(-3.2) = 10.92$

endpoints: none, as domain is $\mathbb{R}$

Example 4 Sketch a graph of $g : [0, 6) \to \mathbb{R}$, where $g(x) = 2x^2 - 8x + 11$

$a = 2, \quad b = -8, \quad c = 11$

x-intercepts: none

$\Delta = (-8)^2 - 4(2)(11) = -24$

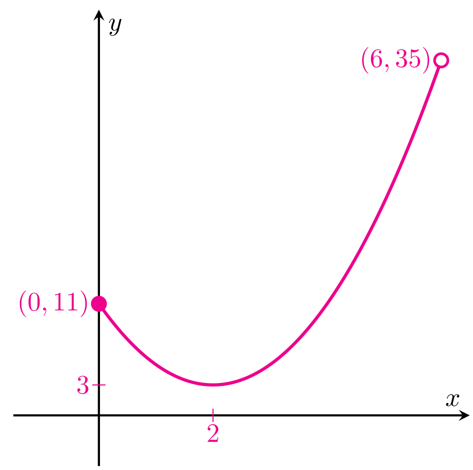
y-intercept: $(0, 11)$

vertex: $(2, 3)$

$h = -\frac{-8}{2(2)} = 2$

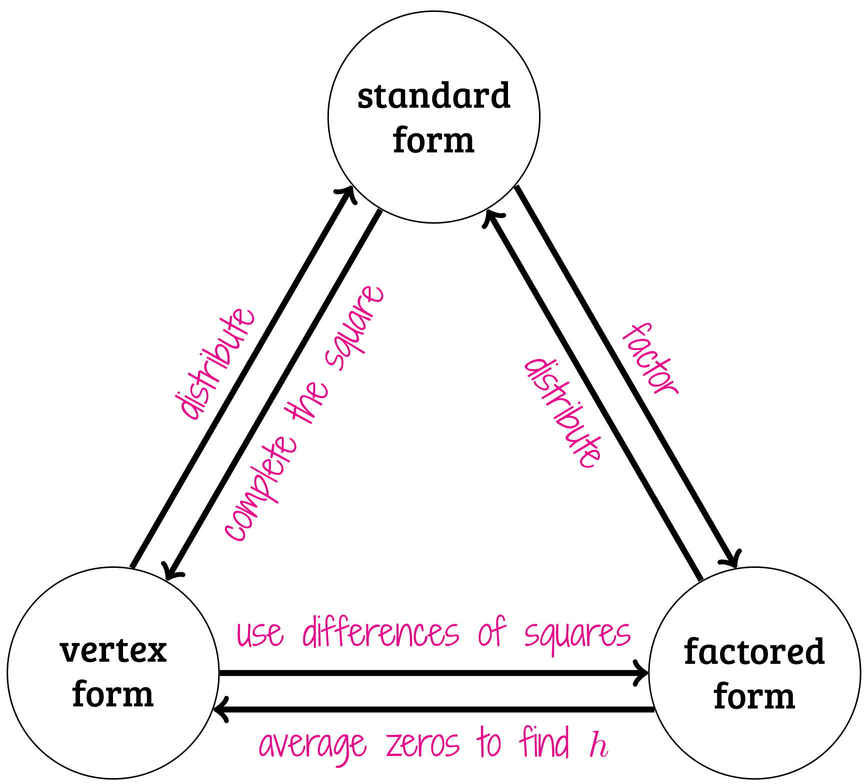
$k = g(2) = 2(2)^2 - 8(2) + 11 = 3$

endpoints: $(0, 11)$ and $(6, 35)$



Converting Quadratics Between Forms

Throughout this chapter we've seen examples of converting between the three forms of quadratic functions. This diagram summarizes those methods.



In practice, if converting between vertex and factored forms, it's often easier to convert to standard form first.

Chapter 4

Further Quadratics

4.1 Complex Numbers 68
4.2 Quadratic Equations with Complex Solutions 72
4.3 Systems Involving Quadratic Equations 74
4.4 Quadratic Regression 77

4.1 Complex Numbers

Recall that some quadratic equations have no real solutions, even if they are something simple, such as

$$x^2 + 1 = 0$$

We can solve equations like this by introducing numbers outside the set of real numbers, known as imaginary numbers.[1]

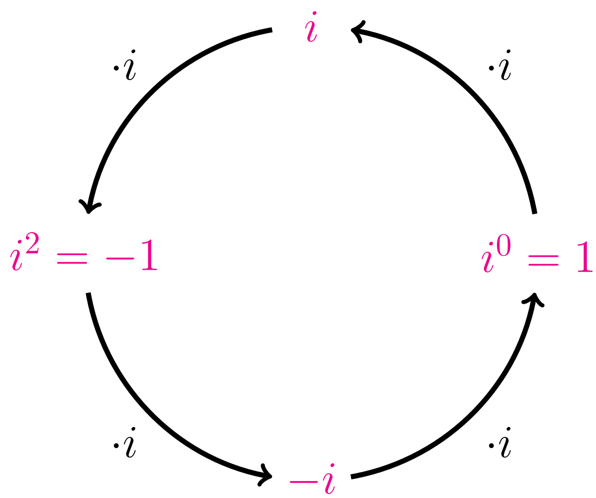
The imaginary unit, denoted by i, is a number defined as having the property

$$i^2 = -1 \qquad \Longrightarrow \qquad \sqrt{-1} = i$$

and is a solution to the equation above.

The powers of i follow a very particular pattern:

i^0		1
i^1		i
i^2		-1
i^3	$i^2 \cdot i = -1 \cdot i$	$-i$
i^4	$i^3 \cdot i = -i \cdot i = -i^2$	1
i^5	$i^4 \cdot i = 1 \cdot i$	i
i^6	$i^5 \cdot i = i \cdot i = i^2$	-1
i^7	$i^6 \cdot i = -1 \cdot i$	$-i$
i^8	$i^7 \cdot i = -i \cdot i = -i^2$	1



Example 1 Evaluate each of the following.

$$\begin{aligned} i^{27} &= (i^4)^6 \cdot i^3 \\ &= 1^6 \cdot (-i) \\ &= -i \end{aligned} \qquad \begin{aligned} i^{394} &= (i^4)^{98} \cdot i^2 \\ &= 1^{98} \cdot (-1) \\ &= -1 \end{aligned} \qquad \begin{aligned} i^{-23} &= (i^4)^{-6} \cdot i^1 \\ &= 1^{-6} \cdot i \\ &= i \end{aligned}$$

[1]Don't let the name fool you! Imaginary numbers may be abstract, but so are all numbers, and that doesn't mean they don't exist. Imaginary numbers have *many* applications in science and engineering. The mathematical terms *real* and *imaginary* are not entirely accurate, but they've been around for so long that we're stuck with them.

An imaginary number is any real number multiplied by i.

A complex number is any number of the form $a + bi$ where a and b are real numbers. Note that if $b = 0$, the resulting complex number is real. Therefore, the real numbers are a subset of the complex numbers.

Typed	Written	Name	Description
$\mathbb{C}$		the complex numbers	The set containing all real and imaginary numbers, and their linear combinations.

For a given complex number, z, the real part is denoted by $\text{Re}(z)$, and the imaginary part is denoted by $\text{Im}(z)$.

Example 2 Find the real and imaginary parts of each of the following.

$z_1 = 3 + 7i$
$\text{Re}(z_1) = 3$
$\text{Im}(z_1) = 7$

$z_2 = -5 + 11i$
$\text{Re}(z_2) = -5$
$\text{Im}(z_2) = 11$

$z_3 = 9 - 13i$
$\text{Re}(z_3) = 9$
$\text{Im}(z_3) = -13$

Adding and Subtracting Complex Numbers

To add and subtract complex numbers, add and subtract the real and imaginary parts of the numbers independently. That is,

$$\text{Re}(z_1 \pm z_2) = \text{Re}(z_1) \pm \text{Re}(z_2) \qquad \text{Im}(z_1 \pm z_2) = \text{Im}(z_1) \pm \text{Im}(z_2)$$

Example 3 Evaluate the following using z_1, z_2 and z_3 above.

$$z_1 + z_2 = (3 - 5) + (7 + 11)i = -2 + 18i$$

$$z_2 + z_3 = (-5 + 9) + (11 - 13)i = 4 - 2i$$

$$z_3 - z_1 = (9 - 3) + (-13 - 7)i = 6 - 20i$$

$$z_1 - z_2 = (3 + 5) + (7 - 11)i = 8 - 4i$$

Multiplying Complex Numbers

Complex numbers can be multiplied using the distributive property as usual, which we can represent using the box method. Don't forget to replace i^2 with -1.

Example 4 Evaluate $(2+5i)(3-7i)$

	2	$+5i$
3	6	$+15i$
$-7i$	$-14i$	35

$$(2+5i)(3-7i) = 41 + i$$

Example 5 Evaluate $(-1-8i)(5-4i)$

	-1	$-8i$
5	-5	$-40i$
$-4i$	$+4i$	-32

$$(-1-8i)(5-4i) = -37 - 36i$$

Complex Conjugates

The conjugate of a complex number is the result of reversing the sign of the imaginary part of the number. The real part is unchanged. Conjugation is denoted by a horizontal bar over the number or variable.

Example 6 Find the conjugate of each of the following.

$z_1 = 3 + 7i$ $\quad$ $z_2 = -5 + 11i$ $\quad$ $z_3 = 9 - 13i$

$\overline{z_1} = 3 - 7i$ $\quad$ $\overline{z_2} = -5 - 11i$ $\quad$ $\overline{z_3} = 9 + 13i$

Example 7 Multiply $z = 3 - 4i$ by its conjugate.

	3	$-4i$
3	9	$-12i$
$+4i$	$+12i$	$+16$

$$\begin{aligned} z\bar{z} &= (3-4i)(3+4i) \\ &= 9 + 12i - 12i + 16 \\ &= 25 \end{aligned}$$

Dividing Complex Numbers

When we divide, the aim is to write the final result in the form $a + bi$, which takes a little more algebraic manipulation than the other operations.

This method relies on the property that the product of a complex number and its conjugate is a real number.

1. Write the division as a fraction.
2. Multiply both the numerator and denominator by the conjugate of the denominator.
3. Evaluate each product.
4. Simplify to the form $a + bi$.

Example 8 Simplify $\dfrac{2}{3+5i}$

$$\frac{2}{3+5i} = \frac{2(3-5i)}{(3+5i)(3-5i)}$$

$$= \frac{6-10i}{34}$$

$$= \tfrac{3}{17} - \tfrac{5}{17}i$$

	3	$+5i$
3	9	$+15i$
$-5i$	$-15i$	$+25$

Example 9 Simplify $\dfrac{3+4i}{5-2i}$

$$\frac{3+4i}{5-2i} = \frac{(3+4i)(5+2i)}{(5-2i)(5+2i)}$$

$$= \frac{7+26i}{29}$$

$$= \tfrac{7}{29} + \tfrac{26}{29}i$$

	3	$+4i$
5	15	$+20i$
$+2i$	$+6i$	-8

	5	$-2i$
5	25	$-10i$
$+2i$	$+10i$	$+4$

4.2 Quadratic Equations with Complex Solutions

Recall that when the discriminant of a quadratic equation, $\Delta = b^2 - 4ac$, is negative, the equation has no real solutions. It turns out that these equations do indeed have solutions.

Theorem

Every quadratic equation $ax^2 + bx + c = 0$ has two solutions (when multiplicity[2] is considered), whose nature is determined by the discriminant $\Delta = b^2 - 4ac$:

1. If $\Delta > 0$, then there are two distinct real solutions.
2. If $\Delta = 0$, then there is one real solution with a multiplicity[2] of two.
3. If $\Delta < 0$, then there are two complex conjugate solutions.

Example 1 Solve each of the following equations with complex solutions.

$$\begin{aligned} x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \pm\sqrt{-9} \\ &= \pm\sqrt{9}\sqrt{-1} \\ &= \pm 3i \end{aligned}$$

$$\begin{aligned} x^2 + 75 &= 0 \\ x^2 &= -75 \\ x &= \pm\sqrt{-75} \\ &= \pm\sqrt{75}\sqrt{-1} \\ &= \pm 5\sqrt{3}i \end{aligned}$$

$$\begin{aligned} (x+4)^2 + 36 &= 0 \\ (x+4)^2 &= -36 \\ x + 4 &= \pm\sqrt{-36} \\ &= \pm 6i \\ x &= -4 \pm 6i \end{aligned}$$

Generally, quadratic equations with complex solutions can be solved in the usual way using completing the square or the quadratic formula.

Example 2 Determine the nature of the solutions of $x^2 = 2x - 5$, then solve it.

$$x^2 - 2x + 5 = 0 \quad \Longrightarrow \quad a = 1, \quad b = -2, \quad c = 5$$

$\Delta = (-2)^2 - 4(1)(5) = -16 < 0 \quad \Longrightarrow$ The solutions are complex conjugates.

$$\begin{aligned} x^2 - 2x + 1 + 4 &= 0 \\ (x-1)^2 &= -4 \\ x - 1 &= \pm\sqrt{4}\sqrt{-1} = \pm 2i \\ x &= 1 \pm 2i \end{aligned}$$

[2] *Multiplicity* will be discussed in more detail in the *Polynomials* chapter.

Example 3 For each equation, determine the nature of the solutions. Verify by solving.

$-3x^2 + 4x - 2 = 0$

$a = -3, \quad b = 4, \quad c = -2$

$\Delta = (4)^2 - 4(-3)(-2) = -8 < 0$

$\implies$ Two complex conjugates solutions.

$$x = \frac{-(4) \pm \sqrt{-8}}{2(-3)} = \frac{-4 \pm 2\sqrt{2}i}{-6} = \tfrac{2}{3} \pm \tfrac{\sqrt{2}}{3}i$$

$4x^2 + 25 = 20x$

$4x^2 - 20x + 25 = 0$

$a = 4, \quad b = -20, \quad c = 25$

$\Delta = (-20)^2 - 4(4)(25) = 0$

$\implies$ One real solution.

$$x = \frac{-(-20) \pm \sqrt{0}}{2(4)} = \frac{20}{8} = \tfrac{5}{2}$$

$3x^2 + 6x = 1$

$3x^2 + 6x - 1 = 0$

$a = 3, \quad b = 6, \quad c = -1$

$\Delta = (6)^2 - 4(3)(-1) = 48 > 0$

$\implies$ Two real solutions.

$$x = \frac{-(6) \pm \sqrt{48}}{2(3)} = \frac{-6 \pm 4\sqrt{3}}{6} = -1 \pm \tfrac{2\sqrt{3}}{3}$$

4.3 Systems Involving Quadratic Equations

Quadratic-Linear Systems

Previously, we've worked with systems consisting of only linear equations. We now have the tools necessary to solve systems when quadratic equations are included as well.

The meaning of a solution to a quadratic-linear system is unchanged. A solution consists of values for each variable which satisfy each equation simultaneously (at the same time.) Because quadratics are involved, there may be zero, one or two real solutions.

As with linear systems, the goal is to algebraically manipulate the system so that all variables except one are eliminated, resulting in a single equation, which can be solved by the usual means.

Don't forget to solve for BOTH variables!

Example 1 Solve the system.

$$\begin{cases} y = x^2 + 6x - 33 & (1) \\ y = 3x - 5 & (2) \end{cases}$$

Equate y from each equation:

$$\begin{aligned} x^2 + 6x - 33 &= 3x - 5 \\ x^2 + 3x - 28 &= 0 \\ (x+7)(x-4) &= 0 \\ x = -7 &\text{ or } x = 4 \end{aligned}$$

Substitute into (2):

$$\begin{aligned} y &= 3(-7) - 5 = -26 \\ y &= 3(4) - 5 = 7 \end{aligned}$$

Solutions: $(-7, -26)$ and $(4, 7)$

Example 2 Solve the system to 2 decimal places.

$$\begin{cases} x + 3y = 6 & (1) \\ \qquad y = x^2 - 5 & (2) \end{cases}$$

Substitute (2) into (1):

$$\begin{aligned} x + 3(x^2 - 5) &= 6 \\ 3x^2 + x - 15 &= 6 \\ 3x^2 + x - 21 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-(1) \pm \sqrt{(1)^2 - 4(3)(-21)}}{2(3)} \\ &= -2.8177 \text{ or } 2.4843 \end{aligned}$$

Substitute into (2):

$$\begin{aligned} y &= (-2.8177)^2 - 5 = 2.94 \\ y &= (2.4843)^2 - 5 = 1.17 \end{aligned}$$

Solutions: $(-2.82, 2.94)$ and $(2.48, 1.17)$

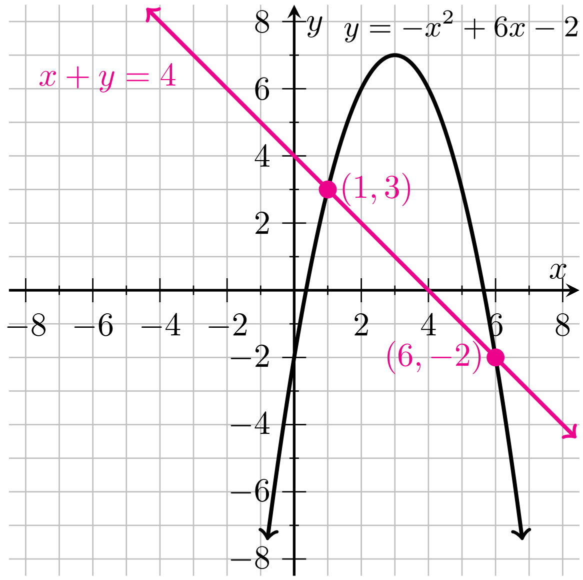
Example 3 Graphically find the solutions to the system

$$\begin{cases} y = -x^2 + 6x - 2 \\ x + y = 4 \end{cases}$$

The curve for $y = -x^2 + 6x - 2$ is already plotted.

Line $x + y = 4$ has intercepts at $(0, 4)$ and $(4, 0)$.

Solutions: $x = 1, y = 3$ and $x = 6, y = -2$.



Example 4 Determine the number of real solutions of the system

$$\begin{cases} y = 5x + 11 \\ y = -x^2 + 2x + 8 \end{cases}$$

$$5x + 11 = -x^2 + 2x + 8$$
$$x^2 + 3x + 3 = 0$$
$$\Delta = b^2 - 4ac$$
$$= 3^2 - 4(1)(3)$$
$$= -3 < 0$$
$\Longrightarrow$ there are no real solutions.

Example 5 Find k such that the system has exactly one solution.

$$\begin{cases} y = -x^2 + 4x - 4 \\ y = kx - 3 \end{cases}$$

$$kx - 3 = -x^2 + 4x - 4$$ Equate y from each equation.
$$x^2 + (k-4)x + 1 = 0$$
$$\Delta = b^2 - 4ac = 0$$ As we want one solution.
$$(k-4)^2 - 4 = 0$$
$$(k-4)^2 = 4$$
$$k - 4 = \pm 2$$
$$k = 4 \pm 2$$
$k = 2$ or $k = 6$

Identifying Quadratics using Linear Systems

Suppose we know that a function f is quadratic, and that $f(3) = 5$. The function can be written in standard form as

$$f(x) = ax^2 + bx + c$$

which, by substituting $x = 3$ and $f(x) = 5$, becomes the equation

$$9a + 3b + c = 5$$

Is it possible to identify $f(x)$ from this equation?

No, because there is only one equation with three unknowns: a, b and c.

Recall that a system in three unknowns requires three equations to be solvable.

Theorem

A quadratic function can be identified if it has known values at three points on the domain.

Example 6 Find the quadratic function f which satisfies $f(3) = 5$, $f(0) = -1$ and $f(4) = 15$.

Let $f(x) = ax^2 + bx + c$, which creates the system:

$$\begin{cases} 9a + 3b + c = 5 \\ c = -1 \\ 16a + 4b + c = 15 \end{cases}$$

$$c = -1 \implies \begin{cases} 9a + 3b - 1 = 5 \\ 16a + 4b - 1 = 15 \end{cases} \implies \begin{cases} 9a + 3b = 6 & (1) \\ 16a + 4b = 16 & (2) \end{cases}$$

Multiplying (1) by -4 and (2) by 3:

$$\begin{cases} -36a - 12b = -24 \\ 48a + 12b = 48 \end{cases} \implies 12a = 24 \implies a = 2$$

Substituting into (1):

$$18 + 3b = 6 \implies 3b = -12 \implies b = -4$$

$$\implies f(x) = 2x^2 - 4x - 1$$

4.4 Quadratic Regression

Recall that regression is the process of fitting a modeling function to a set of data in order to approximate the relationship between variables.

Quadratic regression uses a quadratic function for the model. It is typical to use the standard form of the function. In practice, this means choosing values for a, b and c so that $f(x) = ax^2 + bx + c$ fits the data as well as possible.

The coefficient of determination has the same meaning as for linear regression: it is a measure of how well the regression curve fits the data. For non-linear regression, R^2 has no relation to r.

Example 1 A camera captures the flight of a ball after it is thrown. The frames are analyzed, and the following data is recorded showing the horizontal distance, x, of the ball from where it was thrown versus its vertical height above the ground, y.

x (ft)	1.0	3.0	5.0	7.0	9.0	11.0
y (ft)	7.3	9.6	11.6	13.4	15.1	16.3

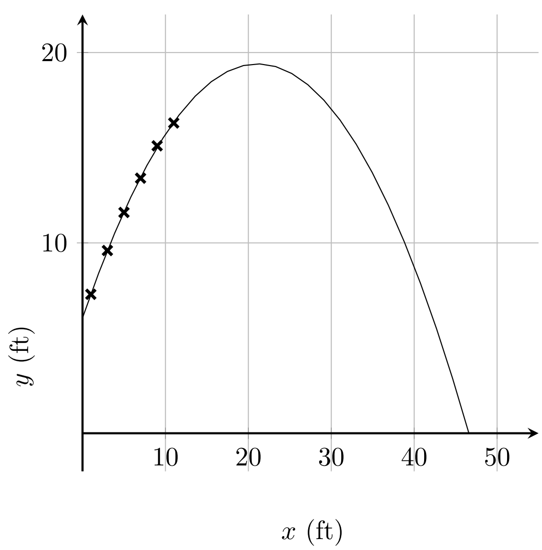
Use quadratic regression to model the flight of the ball.

Using technology,

$a = -0.0299, b = 1.2632, c = 6.0631, R^2 = 0.9998$

$$y = -0.0299x^2 + 1.2632x + 6.0631$$

Once technology is used to perform a regression, it is usually simple to use the same technology to plot the modeling function with the data, and perform further calculations related to the function.



Example 2 Comment on how well the model fits the data.

The value of R^2, which is close to 1, suggests the model is a very good fit. This is supported by a visual inspection of the data and the model.

Example 3 Estimate the height of the ball after it has traveled 6.4 ft.

When $x = 6.4$, we have

$$\begin{aligned} y &= -0.0299(6.4)^2 + 1.2632(6.4) + 6.0631 \\ &= 12.9\,\text{ft} \end{aligned}$$

Example 4 Predict the maximum height of the ball, and the distance it will travel before hitting the ground.

Using technology, the modeling function has a vertex at $(21.116, 19.4)$ and an x-intercept at $(46.584, 0)$.

Maximum height: 19.4 ft. Distance travelled: 46.6 ft.

Note that to answer the previous example, we had to use extrapolation, which may make the prediction unreliable. In this case, physics predicts that a "projectile" (such as the ball in the examples) has a parabolic path, which increases our confidence in our quadratic model, so the predictions seem sensible.

But suppose that someone catches the ball before it hits the ground. Then our prediction of the distance the ball will travel is incorrect. Always be careful using extrapolation, as additional information may be needed to accept or reject our predictions.

Chapter 5

Polynomials

5.1 Polynomial Concepts 80
5.2 Cubic Functions 82
5.3 Special Cubics 84
5.4 Polynomial Division 86
5.5 Factoring Polynomials 90
5.6 Graphs of Polynomial Functions 92

5.1 Polynomial Concepts

A polynomial is an expression which, in standard form, can be written as

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where

- n, and the following decreasing exponents, are integers greater than or equal to zero.
- $a_n, a_{n-1}, \ldots, a_0$ are coefficients (real numbers[1]).
- $a_n \neq 0$.

The largest exponent, n, is called the degree of the polynomial.

The terms of a polynomial are the separate expressions of the form $a_i x^i$. The polynomial is the sum of its terms.

Example 1 Write $P(x) = 9x^2 - 3x^3 - 11 + 12x^5 - 2x + 7x^2 + 5$ in standard form.

$$P(x) = 12x^5 - 3x^3 + 16x^2 - 2x - 6$$

Naming Polynomials by Degree

degree	name	example
0	constant	7
1	linear	$3x - 9$
2	quadratic	$5x^2 + 9x$
3	cubic	$-4x^3 - 7x + 1$
4	quartic	$12x^4 - 8x^2 + 11x$
5	quintic	$-3x^5 + x^3$

If the polynomial has a higher degree, it can be referred to as a nth-degree polynomial.
For example, $5x^9 - x^8 + 6x^7$ is a 9th-degree polynomial.

[1] In general, mathematicians consider polynomials with coefficients of all sorts of number types. For us, they will always be real.

Naming Polynomials by Number of Terms

terms	name	example
1	monomial	$5x^3$
2	binomial	$5x+6$
3	trinomial	$x^5-4x^3+9x^2$

The name polynomial is a generalization of these names, with the prefix poly- meaning any number of terms fits the definition.

Example 2 x^4-7x^2 is a quartic binomial.

Adding and Subtracting Polynomials

To add or subtract polynomials, add or subtract the coefficients of terms with matching exponents.

Example 3 Add $3x^4+7x^3-9x^2+5$ and $-8x^4+5x^3+2x-3$.

$$\begin{array}{rrrrrr} & (\ 3x^4 & +7x^3 & -9x^2 & & +5\ ) \\ + & (-8x^4 & +5x^3 & & +2x & -3\ ) \\ \hline & -5x^4 & +12x^3 & -9x^2 & +2x & +2 \end{array}$$

Example 4 Subtract $5x^4-3x^2+4x-11$ and $x^4-7x^3+9x^2-6$.

$$\begin{array}{rrrrrr} & (\ 5x^4 & & -3x^2 & +4x & -11\ ) \\ - & (\ x^4 & -7x^3 & +9x^2 & & -6\ ) \\ \hline & 4x^4 & +7x^3 & -12x^2 & +4x & -5 \end{array}$$

Multiplying Polynomials

Polynomials are multiplied using the distributive property, which was covered in Sec. 3.3.

Example 5 Distribute $(2x^2-7x)(x^5+3x^3-9x^2)$

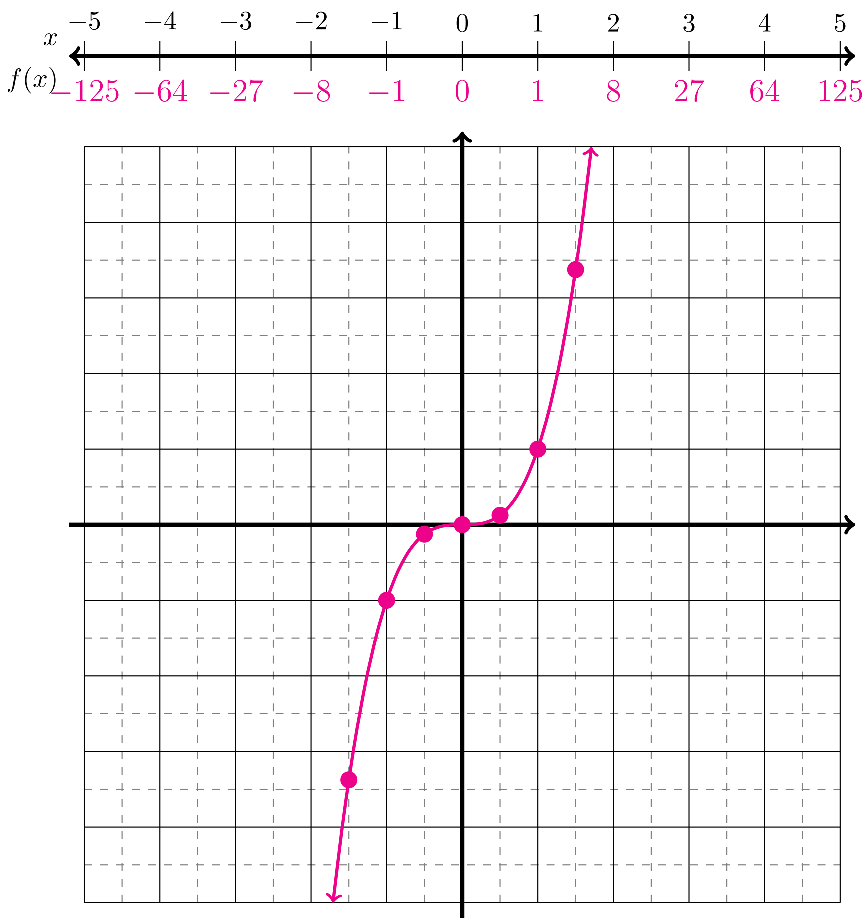
	x^5	$+0x^4$	$+3x^3$	$-9x^2$
$2x^2$	$2x^7$	$0x^6$	$+6x^5$	$-18x^4$
$-7x$	$-7x^6$	$0x^5$	$-21x^4$	$+63x^3$

$$(2x^2-7x)(x^5+3x^3-9x^2)=2x^7-7x^6+6x^5-39x^4+63x^3$$

5.2 Cubic Functions

Graphing polynomials becomes more difficult as their degree increases past two. An exception is functions resulting from transformations applied to the parent cubic function.

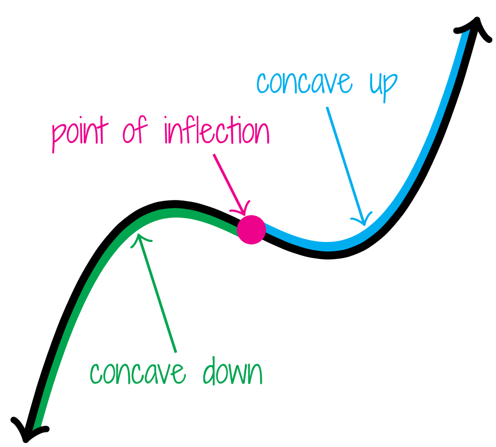
parent function	$f(x) = x^3$
domain	$\mathbb{R}$
range	$\mathbb{R}$
relation type	one-to-one
x-intercept	$(0, 0)$
y-intercept	$(0, 0)$
point of inflection	$(0, 0)$



The graphs of cubic functions have a point of inflection, which is a point where the curvature changes direction.

In the case of the parent function $f(x) = x^3$, the curve changes from concave down to concave up at $(0, 0)$.

Note that while the parent cubic function is one-to-one, this is not true of all cubic functions, including the one shown in the diagram here.



Graphing Cubic Functions Using Transformations

By applying <u>transformations</u> to the cubic parent function, we get the form $f(x) = A(x-h)^3 + k$. Only a tiny subset of cubic functions can be written in this form. A sketch of this type of cubic function should include:

shape of curve	cubic curve with enough points to show stretch/compression
point of inflection	(h, k), using translation of parent function to identify
x-intercept	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$
endpoints	evaluate the function at the bounds of the domain

Example 1 Sketch $f(x) = \frac{1}{2}(x-3)^3 + 4$.

Orientation: Upright Point of Inflection: $(3, 4)$

x-intercept: $(1, 0)$

$\frac{1}{2}(x-3)^3 + 4 = 0$

$\frac{1}{2}(x-3)^3 = -4$

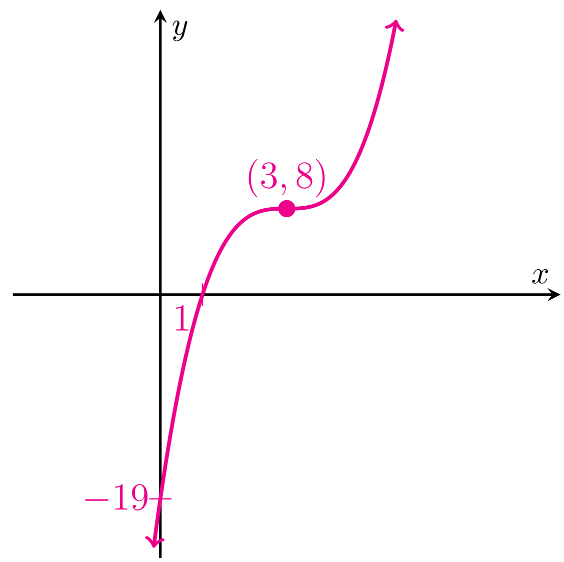
$(x-3)^3 = -8$

$x - 3 = \sqrt[3]{-8} = -2 \implies x = 1$

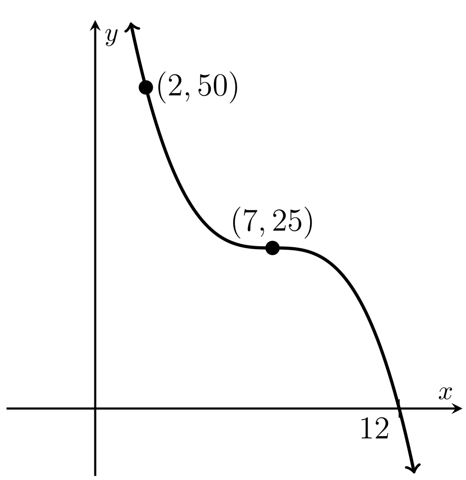
y-intercept: $(0, -9.5)$

as $f(0) = \frac{1}{2}(-3)^3 + 4 = -9.5$

endpoints: none, as domain is $\mathbb{R}$



Example 2 Find the function g represented by the following graph.



Point of inflection: $(7, 25)$

$h = 7, k = 25 \implies g(x) = A(x-7)^3 + 25$

Other point: $(2, 50)$

$g(2) = (-5)^3 A + 25 = 50$

$-125A = 25 \implies A = -\frac{1}{5}$

$$g(x) = -\frac{1}{5}(x-7)^3 + 25$$

5.3 Special Cubics

Theorem: Perfect Cubes

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$$

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$$

Proof

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a(a^2+2ab+b^2) + b(a^2+2ab+b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

	a	$+b$
a^2	a^3	$+a^2b$
$+2ab$	$+2a^2b$	$+2ab^2$
$+b^2$	$+ab^2$	$+b^3$

Replace b with $-b$ to obtain the second result. $\blacksquare$

Theorem: Sums and Differences of Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Proof

$$\begin{aligned}&(a+b)(a^2 - ab + b^2) \\ &\quad = a(a^2 - ab + b^2) + b(a^2 - ab + b^2) \\ &\quad = a^3 \cancel{- a^2b} \cancel{+ ab^2} \cancel{+ a^2b} \cancel{- ab^2} + b^3 \\ &\quad = a^3 + b^3\end{aligned}$$

	a	$+b$
a^2	a^3	a^2b
$-ab$	$-a^2b$	$-ab^2$
$+b^2$	$+ab^2$	$+b^3$

Replace b with $-b$ to obtain the second result. $\blacksquare$

As with the special quadratics in section 3.4, we can use these rules to quickly distribute and factor certain expressions.

Example 1 Distribute $(x-5)^3$

Using $a=x$ and $b=5$,

$$(x-5)^3 = x^3 - 15x^2 + 75x - 125$$

Example 2 Distribute $(x+4)(x^2-4x+16)$

Using $a=x$ and $b=4$,

$$(x+4)(x^2-4x+16) = x^3 + 64$$

Example 3 Distribute $(3x+7)^3$

Using $a=3x$ and $b=7$,

$$\begin{aligned}(3x+7)^3 &= 3^3x^3 + 3\cdot 3^2\cdot 7x^2 + 3\cdot 3\cdot 7^2x + 7^3\\ &= 27x^3 + 189x^2 + 441x + 343\end{aligned}$$

Example 4 Factor $x^3 - 1331$

Using $a=x$ and $b=11$,

$$x^3 - 1331 = (x-11)(x^2-11x+121)$$

Example 5 Factor $x^3 + 12x^2 + 48x + 64$

Using $a=x$ and $b=4$,

$$x^3 + 12x^2 + 48x + 64 = (x+4)^3$$

Example 6 Factor $729x^3 - 512$

Using $a=9x$ and $b=8$,

$$729x^3 - 512 = (9x-8)(81x^2+72x+64)$$

Some expressions can be factored by combining these rules with others we've already learned.

Example 7 Factor $2x^8 - 1458x^2$

$$\begin{aligned}2x^8 - 1458x^2 &= 2x^2(x^6 - 729) && \text{using GCF}\\ &= 2x^2(a^2 - 27^2) && \text{where } a = x^3\\ &= 2x^2(a-27)(a+27) && \text{using difference of squares}\\ &= 2x^2(x^3-27)(x^3+27)\\ &= 2x^2(x-3)(x^2+3x+9)(x+3)(x^2-3x+9)\end{aligned}$$

using diff. and sum of cubes

5.4 Polynomial Division

Recall from elementary school, before you learned decimals and fractions, that division of integers results in a remainder when the division isn't exact.

Example 1

$19 \div 7 = 2$ R 5 because $19 = 7 \cdot 2 + 5$

$35 \div 8 = 4$ R 3 because $35 = 8 \cdot 4 + 3$

$63 \div 11 = 5$ R 8 because $63 = 11 \cdot 5 + 8$

Note that the remainder will always be smaller than the divisor. The part of the result which is not the remainder is called the quotient.

Polynomials, as it turns out, are divided in a manner very similar to integers.[2]

Example 2 Verify that when $P(x) = x^4 - x^3 - 13x^2 + 28x - 9$ is divided by $x - 3$, the quotient is $Q(x) = x^3 + 2x^2 - 7x + 7$ and the remainder is 12.

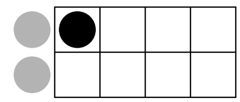
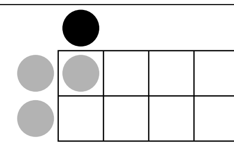
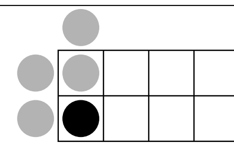
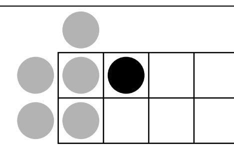
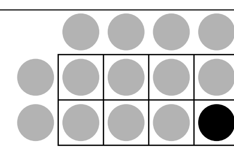
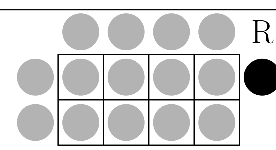
$$\begin{aligned}(x-3)\cdot Q(x) + 12 &= (x-3)(x^3 + 2x^2 - 7x + 7) + 12 \\ &= x^4 - x^3 - 13x^2 + 28x - 21 + 12 \\ &= x^4 - x^3 - 13x^2 + 28x - 9 \\ &= P(x), \text{ as required.}\end{aligned}$$

	x^3	$+2x^2$	$-7x$	$+7$
x	x^4	$+2x^3$	$-7x^2$	$+7x$
-3	$-3x^3$	$-6x^2$	$+21x$	-21

The goal of polynomial division is to find the quotient and the remainder. There are several methods that can be used, but we will use a variation of the box method as we are already familiar with it.

[2] This isn't just a coincidence as it seems to be. Mathematicians actually consider the set of integers and the set of polynomials to have the same underlying algebraic structure.

In the final result, the divisor is placed along the left-hand side of the box grid, and the quotient is placed along the top. The original polynomial is *mostly* contained within the grid, but won't fit perfectly if there is a remainder.

Step 1: Construct the box grid with the divisor along the left-hand side.	
Step 2: Place the first term of the original polynomial in the top-left cell.	
Step 3: Remembering that the usual multiplication rules for the box method apply, complete the entry above the last entry.	
Step 4: Use multiplication to complete the column.	
Step 5: Complete the next cell in the top row so that its diagonal completes the next term in the original polynomial.	
Step 6: Repeat steps 3 to 5 until the box grid is complete.	
Step 6: Add a remainder so that the constant term of the polynomial is complete.	

Example 3 Divide $P(x) = x^3 - 2x^2 - 21x + 7$ by $x + 4$.

	x^2	$-6x$	$+3$	R
x	x^3	$-6x^2$	$+3x$	-5
$+4$	$+4x^2$	$-24x$	$+12$	

$$P(x) = (x+4)(x^2 - 6x + 3) - 5$$

or equivalently

$$\frac{P(x)}{x+4} = x^2 - 6x + 3 - \frac{5}{x+4}$$

Example 4 Divide $P(x) = 4x^3 - 6x^2 + 8$ by $x - 2$.

	$4x^2$	$+2x$	$+4$	R
x	$4x^3$	$+2x^2$	$+4x$	$+16$
-2	$-8x^2$	$-4x$	-8	

$$P(x) = (x-2)(4x^2+2x+4)+16$$

or equivalently

$$\frac{P(x)}{x-2} = 4x^2+2x+4+\frac{16}{x-2}$$

Example 5 Divide $x^4 + x^3 - 17x^2 - 42x - 66$ by $x^2 + 3x + 4$.

Let $P(x) = x^4 + x^3 - 17x^2 - 42x - 66$

	x^2	$-2x$	-15	R
x^2	x^4	$-2x^3$	$-15x^2$	$+11x$
$+3x$	$+3x^3$	$-6x^2$	$-45x$	-6
$+4$	$+4x^2$	$-8x$	-60	

$$P(x) = (x^2+3x+4)(x^2-2x-15)+11x-6$$

or equivalently

$$\frac{P(x)}{x^2+3x+4} = x^2-2x-15+\frac{11x-6}{x^2+3x+4}$$

The Remainder Theorem

Recall that in integer division, the _remainder_ is always less than the _divisor_.

A related idea for polynomials is described by the following theorem.

Theorem

In _polynomial division_, if there is a _remainder_, its _degree_ is always less than the _degree_ of the _divisor_.

If the _divisor_ is _linear_, then the _remainder_ must be a _constant_.

We can easily confirm that this is true for the examples above. In the particular case of a linear divisor, the following theorem is very important:

The Remainder Theorem

Suppose a polynomial, $P(x)$, is divided by a linear binomial, $x - a$.
Then the remainder is equal to $P(a)$.

Proof

Let $Q(x)$ be the quotient, and let R be the remainder.

$$\begin{aligned} P(x) &= (x-a)\cdot Q(x) + R \\ P(a) &= (a-a)\cdot Q(a) + R \\ &= \cancel{0\cdot Q(a)} + R \\ &= R \qquad \blacksquare \end{aligned}$$

Example 6 Confirm the remainder from example 3, dividing $P(x) = x^3 - 2x^2 - 21x + 7$ by $x + 4$.

$$P(-4) = (-4)^3 - 2(-4)^2 - 21(-4) + 7 = -5$$

Example 7 Confirm the remainder from example 4, dividing $P(x) = 4x^3 - 6x^2 + 8$ by $x - 2$.

$$P(2) = 4(2)^3 - 6(2)^2 + 8 = 16$$

If the linear divisor is not monic, then we can use this updated version of the theorem.

Generalized Remainder Theorem

Suppose a polynomial, $P(x)$, is divided by a linear binomial which equals zero when $x = a$.
Then the remainder is equal to $P(a)$.

Example 8 Suppose $P(x) = 2x^3 - x^2 + kx + 27$ is divided by $2x - 3$, and the remainder is 9. Find the value of k.

$2x - 3 = 0$ when $x = \frac{3}{2}$

$$\begin{aligned} P\left(\tfrac{3}{2}\right) &= 2\left(\tfrac{3}{2}\right)^3 - \left(\tfrac{3}{2}\right)^2 + k\left(\tfrac{3}{2}\right) + 27 \\ &= \tfrac{3}{2}k + \tfrac{63}{2} = 9 \\ \tfrac{3}{2}k &= -\tfrac{45}{2} \\ k &= -15 \end{aligned}$$

5.5 Factoring Polynomials

Suppose that a polynomial $P(x)$ is divided by a particular divisor $x - a$, and that the result is a quotient $Q(x)$ with no remainder. This means we can write the statement

$$\begin{aligned} P(x) &= (x-a)Q(x) + \cancelto{0}{R} \\ &= (x-a)Q(x) \end{aligned}$$

which means that $x - a$ is a factor of $P(x)$.

The following is a special case of the Remainder Theorem, when there is no remainder.

The Factor Theorem

$x - a$ is a factor of the polynomial $P(x)$

iff (if and only if) $P(a) = 0$.

This suggests a method we can use to factor the polynomial $P(x)$:

Step 1: Find a value a for which $P(a) = 0$, which means $x - a$ is a factor.

Step 2: Divide $P(x)$ by $x - a$.

Step 3: Continue by factoring the resulting quotient.

Example 1 Factor $P(x) = x^3 - 21x + 20$.

By trying different values of $P(a)$, we get

$P(1) = (1)^3 - 21(1) + 20 = 0$

$\implies x - 1$ is a factor.

$$\begin{aligned} P(x) &= x^3 - 21x + 20 \\ &= (x-1)(x^2 + x - 20) \\ &= (x-1)(x-4)(x+5) \end{aligned}$$

	x^2	$+x$	-20
x	x^3	$+x^2$	$-20x$
-1	$-x^2$	$-x$	$+20$

Example 2 Solve $2x^3 - 7x^2 - 8x + 28 = 0$

Let $P(x) = 2x^3 - 7x^2 - 8x + 28$

$P(2) = 2(2)^3 - 7(2)^2 - 8(2) + 28 = 0$

$\implies$ $x - 2$ is a factor.

$$\begin{aligned} P(x) &= 2x^3 - 7x^2 - 8x + 28 \\ &= (x-2)(2x^2 - 3x - 14) \\ &= (x-2)(2x-7)(x+2) \\ &= 0 \end{aligned}$$

$x - 2 = 0$ or $2x - 7 = 0$ or $x + 2 = 0$

$x = 2$ or $x = \frac{7}{2}$ or $x = -2$

	$2x^2$	$-3x$	-14
x	$2x^3$	$-3x^2$	$-14x$
-2	$-4x^2$	$+6x$	$+28$

	x	$+2$
$2x$	$2x^2$	$+4x$
-7	$-7x$	-14

Example 3 Factor $P(x) = x^5 - 5x^4 - 25x^3 + 65x^2 + 84x$

$$\begin{aligned} P(x) &= x^5 - 5x^4 - 25x^3 + 65x^2 + 84x \\ &= x(\underbrace{x^4 - 5x^3 - 25x^2 + 65x + 84}_{Q(x)}) \end{aligned}$$

$Q(3) = (3)^4 - 5(3)^3 - 25(3)^2 + 65(3) + 84 = 0$

$\implies$ $x - 3$ is a factor of $Q(x)$.

$$\begin{aligned} P(x) &= xQ(x) \\ &= x(x-3)(\underbrace{x^3 - 2x^2 - 31x - 28}_{R(x)}) \end{aligned}$$

$R(-1) = (-1)^3 - 2(-1)^2 - 31(-1) - 28 = 0$

$\implies$ $x + 1$ is a factor of $R(x)$.

$$\begin{aligned} P(x) &= x(x-3)R(x) \\ &= x(x-3)(x+1)(x^2 - 3x - 28) \\ &= x(x-3)(x+1)(x-7)(x+4) \end{aligned}$$

	x^3	$-2x^2$	$-31x$	-28
x	x^4	$-2x^3$	$-31x^2$	$-28x$
-3	$-3x^3$	$+6x^2$	$+93x$	$+84$

	x^2	$-3x$	-28
x	x^3	$-3x^2$	$-28x$
$+1$	$+x^2$	$-3x$	-28

5.6 Graphs of Polynomial Functions

Recall that a polynomial is a type of expression. If it is treated as a function, then it is called a polynomial function.

When analyzing the graphs of polynomial functions, we'll need to think about how the function behaves in two different ways:

- locally, which means we only consider the immediate vicinity (close to) the point we're interested in; and
- globally, which means we consider the function over its entire domain.

Zeros, x-Intercepts and Multiplicity

For a polynomial function, as with all functions, the x-intercepts of its graph correspond to the zeros of the function, which are the input values which cause the output values to equal zero.

Example 1 Find the zeros of $f(x) = (x+1)^2(x-1)^3(x-2)$, and find the x-intercepts of its graph.

$f(x) = 0 \implies x = -1$ or $x = 1$ or $x = 2$.

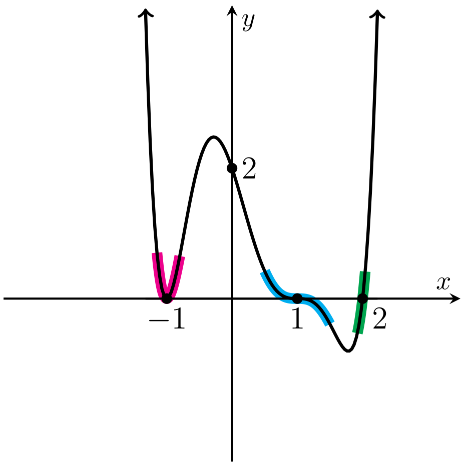
Zeros are -1, 1 and 2. x-intercepts are (-1,0), (1,0), (2,0).

How many zeros are there in this example? If we count them, the simple answer is three. If we're being more precise, we would say this is the number of distinct zeros.

But that's not the only way to count. Note that 1 is a zero because $(x-1)$ is a factor of the polynomial. But it's not a factor just once, but three times. So we can say that 1 is a zero with multiplicity 3. When we count the zeros with multiplicity, there are six.

If a zero has multiplicity	1	2	3
the function behaves locally like it is	linear	quadratic	cubic
and the x-intercept is a	simple intercept	vertex	point of inflection

The y-intercept is found as in any function, at the point $(0, f(0))$.



Example 2 Identify the zeros and their multiplicity of the polynomial function f shown in the graph.

$(-1, 0)$ is a vertex.

$\Longrightarrow$ $x = -1$ is a zero with multiplicity 2.

$(1, 0)$ is a point of inflection.

$\Longrightarrow$ $x = 1$ is a zero with multiplicity 3.

$(2, 0)$ is a simple intercept.

$\Longrightarrow$ $x = 2$ is a zero with multiplicity 1.

Note that this is the same function as in example 1.

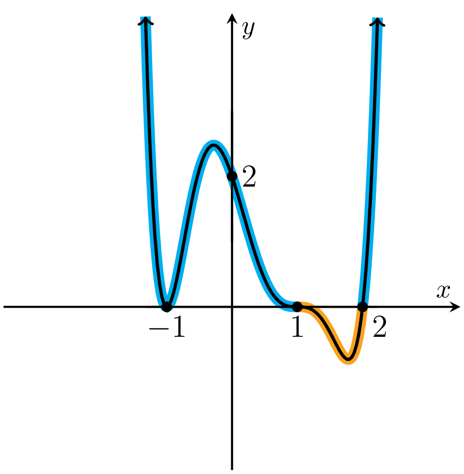
Positive and Negative Intervals

A positive interval is an interval of the domain on which the value of the function is positive, and its graph is above the x-axis.

A negative interval is an interval of the domain on which the value of the function is negative, and its graph is below the x-axis.

Keep in mind that a function's value is zero at its zeros (by definition), and so is neither positive or negative.

If a polynomial function changes sign, it will be at a zero, but not every zero causes a change in sign.



Example 3 Identify the positive and negative intervals for the polynomial function f shown in the graph.

f is positive on the interval

$$(-\infty, -1) \cup (-1, 1) \cup (2, \infty)$$

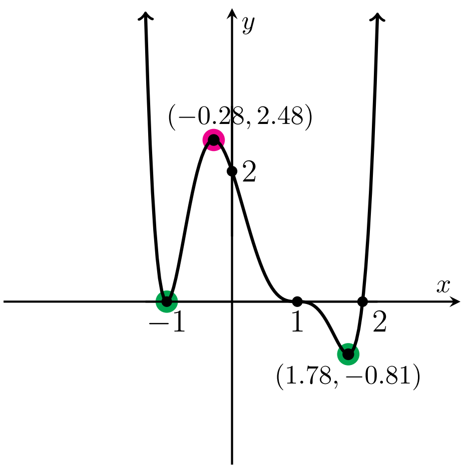
f is negative on the interval

$$(1, 2)$$

Minima and Maxima

A local maximum of a function is a point at which the function has a *greater* value than any points nearby. A local minimum of a function is a point at which the function has a *lesser* value than any nearby points nearby. For polynomial functions, these points occur at vertices.

The global maximum of a function is the point at which the function has a greater value than at every other point in the domain. If it exists, it corresponds with either a local maximum or an endpoint. Similarly, the global minimum has a value less than every other point and, if it exists, corresponds with a local minimum or an endpoint.



Example 4 Identify the (approximate) local and global maxima and minima for the polynomial function f shown in the graph.

f has a local maximum at $(-0.28, 4.48)$

and has no global maximum.

f has local minima at $(-1, 0), (1.78, -0.81)$

and has its global minimum at $(1.78, -0.81)$.

Domain and Range

Polynomials can be evaluated for every real number, so the implied domain of a polynomial function is $\mathbb{R}$. If a graph shows endpoints, however, the domain has been restricted.

Knowing the global maximum and/or minimum, if they exist, will typically allow us to find the range.

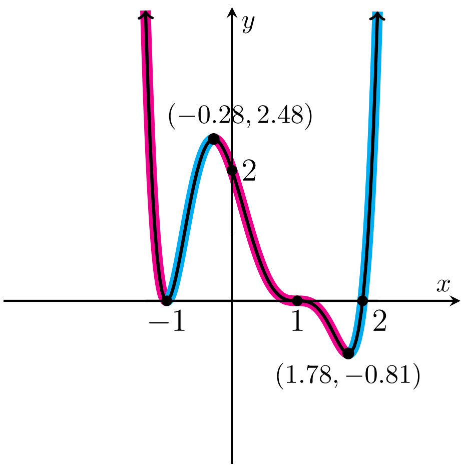
Example 5 State the range of the function above.

$$(-0.81, \infty)$$

Increasing and Decreasing

f is said to be increasing if $f(x)$ increases as x increases, which implies positive slope.

f is said to be decreasing if $f(x)$ decreases as x increases, which implies negative slope.



Example 6 Identify the increasing and decreasing intervals for the polynomial function f shown in the graph.

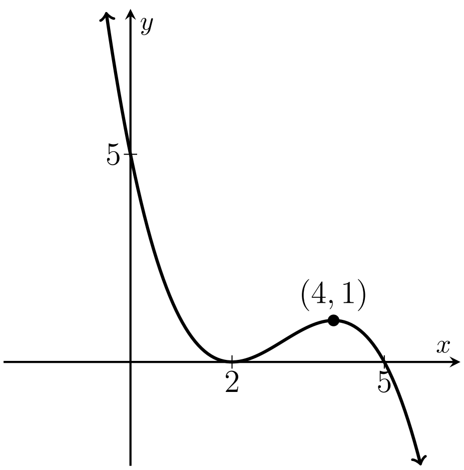
f is increasing on the interval

$$(-1, -0.28) \cup (1.78, \infty)$$

f is decreasing on the interval

$$(-\infty, -1) \cup (-0.28, 1.78)$$

Example 7 Find a polynomial function g to fit the following graph.



The zeros of the function are

- 2 with multiplicity 2
- 5 with multiplicity 1

So a candidate for the function is

$$y = (x-2)^2(x-5)$$

But if $x = 0$, then $y = (-2)^2(-5) = -20$, which doesn't match the y-intercept.

We can use a reflection and compression to change the y-intercept without changing the x-intercepts.

$$g(x) = -\frac{1}{4}(x-2)^2(x-5)$$

Technology can be used to verify that this is the correct function.

Chapter 6

Rational Expressions and Functions

6.1 Simplifying Rational Expressions 98
6.2 Adding and Subtracting Rational Expressions 101
6.3 Complex Fractions 104
6.4 Rational Equations 106
6.5 Simple Rational Functions 108
6.6 Functions with Quadratic Denominators 112

6.1 Simplifying Rational Expressions

Recall that a rational number is a number which can be written in the form of a fraction, where the numerator and denominator are both integers.

Examples	**Non-examples**
$-\frac{5}{6}$ $\quad \frac{2}{3}$ $\quad 7 = \frac{7}{1}$ $\quad 0.75 = \frac{3}{4}$ $\quad \sqrt{25} = \frac{5}{1}$	$\sqrt{11}$ $\quad \pi$ $\quad \sqrt[3]{14}$ $\quad i$ $\quad \sqrt[4]{19}$

Similarly, a rational expression is an expression which can be written in the form of a fraction, where the numerator and denominator are both polynomials.

Examples	**Non-examples**
$\frac{1}{x}$ $\quad \frac{x+3}{x-4}$ $\quad \frac{x^3+2x^2-4x+5}{2x^2-3x+2}$	$\frac{x-7}{\sqrt{x}}$ $\quad \frac{2^x+5}{x-2}$ $\quad \frac{x+1}{\log_2(x)}$

Also recall that any factor (with a key exception) divided by itself is equal to 1. You should be familiar with using this property to simplify fractions.

Examples $\quad \frac{9}{6} = \frac{3 \cdot \cancel{3}}{2 \cdot \cancel{3}} = \frac{3}{2} \qquad \frac{50}{60} = \frac{5 \cdot \cancel{10}}{6 \cdot \cancel{10}} = \frac{5}{6}$

We can use the same property to simplify rational expressions.

Example 1 Simplify $\frac{(x+2)(x-5)}{x-5}$

$$\frac{(x+2)\cancel{(x-5)}}{\cancel{x-5}} = x+2$$

However, if the value being divided by itself is zero, then the expression cannot be simplified like this. Our example has this issue when $x = 5$. If this is the case, the original expression and the simplified version are not equivalent.

When $x = 5$, $\frac{(x+2)(x-5)}{x-5}$ is undefined, but $x+2=7$.

The solution to this problem is to exclude $x = 5$ from our simplification. We call this an excluded value, and we write the result as

$$\frac{(x+2)(x-5)}{x-5} = x+2, \qquad x \neq 5$$

Example 2 Simplify:

$$\frac{12x^3}{3x} = \frac{4 \cdot \cancel{3} \cdot x^2 \cdot \cancel{x}}{\cancel{3} \cdot \cancel{x}}$$
$$= 4x^2 \qquad x \neq 0$$

Example 3 Simplify:

$$\frac{(x-5)\cancel{(x+3)}\cancel{(x-6)}}{\cancel{(x-6)}\cancel{(x+3)}(x+5)} = \frac{x-5}{x+5}$$
$$x \neq 6, -3$$

Example 4 Simplify:

$$\frac{4-x^2}{x^2+x-6} = \frac{(2+x)(2-x)}{(x-2)(x+3)}$$
$$= \frac{-(2+x)\cancel{(x-2)}}{\cancel{(x-2)}(x+3)}$$
$$= -\frac{x+2}{x+3}, \qquad x \neq 2$$

Example 5 Simplify:

$$\frac{x^3+125}{x^3+15x^2+75x+125}$$
$$= \frac{\cancel{(x+5)}(x^2-5x+25)}{(x+5)^{\cancel{3}2}}$$
$$= \frac{x^2-5x+25}{(x+5)^2}$$

An Error to Avoid

Remember that only factors can be eliminated by dividing, not terms. With an expression like the one in example 4, a common error is to do the following.

Don't do this: $\dfrac{\cancel{x^2}+5x+6}{\cancel{x^2}+x-6} = \dfrac{5x+6}{x-6}$ Seriously, DO NOT DO THIS!

This is because the inverse operation of division is multiplication, not addition or subtraction.

Multiplying and Dividing Rational Expressions

Recall that fractions can be multiplied by multiplying the numerators and multiplying the denominators.

Example $\dfrac{3}{5} \cdot \dfrac{11}{6} = \dfrac{3 \cdot 11}{5 \cdot 6} = \dfrac{33}{30} = \dfrac{11}{10}$

Also, recall that dividing by a fraction is the same as multiplying by its reciprocal.

Example $\dfrac{4}{7} \div \dfrac{8}{9} = \dfrac{4}{7} \cdot \dfrac{9}{8} = \dfrac{36}{56} = \dfrac{9}{14}$

Note that in these examples, some simplifying could have been done at the start.

$$\frac{3}{5} \cdot \frac{11}{6} = \frac{1}{5} \cdot \frac{11}{2} = \frac{11}{10} \qquad \frac{4}{7} \div \frac{8}{9} = \frac{4}{7} \cdot \frac{9}{8} = \frac{1}{7} \cdot \frac{9}{2} = \frac{9}{14}$$

The same methods can be used to multiply and divide rational expressions. It is always a good idea to factor and simplify whenever possible.

Example 6 Simplify:

$$\frac{x^2-2x-8}{x+3}\cdot\frac{x+3}{x^2+4x-32}=\frac{\cancel{(x-4)}(x+2)}{\cancel{x+3}}\cdot\frac{\cancel{x+3}}{\cancel{(x-4)}(x+8)}$$

$$=\frac{x+2}{x+8}\qquad x\neq 4,-3$$

Example 7 Simplify:

$$\frac{x^2+12x+35}{3x^2+x-10}\cdot\frac{x^2+9x+14}{x+5}=\frac{(x+7)\cancel{(x+5)}}{(3x-5)\cancel{(x+2)}}\cdot\frac{(x+7)\cancel{(x+2)}}{\cancel{x+5}}$$

$$=\frac{(x+7)^2}{3x-5}\qquad x\neq -5,-2$$

Example 8 Simplify:

$$\frac{x^2+7x-30}{x-4}\div(x^2+6x-40)=\frac{(x-3)(x+10)}{x-4}\div[(x-4)(x+10)]$$

$$=\frac{(x-3)\cancel{(x+10)}}{x-4}\cdot\frac{1}{(x-4)\cancel{(x+10)}}$$

$$=\frac{x-3}{x-4}\cdot\frac{1}{x-4}$$

$$=\frac{x-3}{(x-4)^2}\qquad x\neq -10$$

Example 9 Simplify:

$$\frac{x^2+7x+10}{x^2-x-6}\div\frac{x^2+6x+5}{x^2+x-12}=\frac{(x+2)(x+5)}{(x+2)(x-3)}\div\frac{(x+5)(x+1)}{(x-3)(x+4)}$$

$$=\frac{\cancel{(x+2)}\cancel{(x+5)}}{\cancel{(x+2)}\cancel{(x-3)}}\cdot\frac{\cancel{(x-3)}(x+4)}{\cancel{(x+5)}(x+1)}$$

$$=\frac{x+4}{x+1}\qquad x\neq -5,-4,-2,3$$

In the last example, there's an extra excluded value at -4. The factor $x+4$ is not eliminated, but it is originally in a denominator. If $x=-4$, the original expression is undefined.

6.2 Adding and Subtracting Rational Expressions

Recall that fractions can be added or subtracted if they have the same denominator.

Examples

$$\frac{2}{5}+\frac{7}{10}=\frac{4}{10}+\frac{7}{10}=\frac{11}{10} \qquad \frac{3}{4}-\frac{1}{6}=\frac{9}{12}-\frac{2}{12}=\frac{7}{12}$$

Similarly, rational expressions can be added or subtracted if they have the same denominator.

Example 1 Simplify:

$$\begin{aligned}\frac{x^2+8x}{x^2+7x+12}-\frac{10x+24}{x^2+7x+12} &= \frac{x^2-2x-24}{x^2+7x+12}\\ &= \frac{(x-6)\cancel{(x+4)}}{\cancel{(x+4)}(x+3)}\\ &= \frac{x-6}{x+3} \qquad x \neq -4\end{aligned}$$

Example 2 Simplify:

$$\begin{aligned}\frac{x-12}{x-3}+\frac{4x+15}{x^2-3x} &= \frac{x(x-12)}{x(x-3)}+\frac{4x+15}{x^2-3x}\\ &= \frac{x^2-12x}{x^2-3x}+\frac{4x+15}{x^2-3x}\\ &= \frac{x^2-8x+15}{x^2-3x}\\ &= \frac{\cancel{(x-3)}(x-5)}{x\cancel{(x-3)}}\\ &= \frac{x-5}{x} \qquad x \neq 3\end{aligned}$$

Finding the Lowest Common Multiple

The lowest common multiple of two (or more) expressions is the simplest expression which is a multiple of each given expression.

To find the LCM, find the simpliest multiplier for each expression so that each has the same product, which is the LCM.

Example 3 Find the lowest common multiple of $5x$, $10x^2y$ and $15y^3$.

$$5x \cdot 6xy^3$$
$$10x^2y \cdot 3y^2$$
$$15y^3 \cdot 2x^2$$
$$\text{LCM} = 30x^2y^3$$

Example 4 Find the lowest common multiple of $(x-6)^2$ and $(x-6)(x+8)$.

$$(x-6)^2 \cdot (x+8)$$
$$(x-6)(x+8) \cdot (x-6)$$
$$\text{LCM} = (x-6)^2(x+8)$$

Example 5 Find the lowest common multiple of $x(x-2)$ and $(x-2)(x+5)$.

$$x(x-2) \cdot (x+5)$$
$$(x-2)(x+5) \cdot x$$
$$\text{LCM} = x(x-2)(x+5)$$

Example 6 Find the lowest common multiple of $x^2+9x+20$ and $x^2-2x-35$.

$$(x+4)(x+5) \cdot (x-7)$$
$$(x-7)(x+5) \cdot (x+4)$$
$$\text{LCM} = (x+4)(x+5)(x-7)$$

Adding or Subtracting with Different Denominators

If the denominators are different, we look to find the LCM of the denominators, and make that the common denominator.

It is best practice to simplify and factor the resulting numerator, in case the expression can simplify further.

Example 7 Simplify:

$$\frac{x}{x+1} - \frac{4}{x+4} = \frac{x(x+4)}{(x+1)(x+4)} - \frac{4(x+1)}{(x+7)(x+1)}$$
$$= \frac{x^2+4x}{(x+1)(x+4)} - \frac{4x+4}{(x+4)(x+1)}$$
$$= \frac{x^2-4}{(x+1)(x+4)}$$
$$= \frac{(x+2)(x-2)}{(x+1)(x+4)}$$

Example 8 Simplify:

$$\frac{5}{x^2+9x+14}+\frac{x}{x^2+6x+8}=\frac{5}{(x+2)(x+7)}+\frac{x}{(x+2)(x+4)}$$

$$=\frac{5(x+4)}{(x+2)(x+7)(x+4)}+\frac{x(x+7)}{(x+2)(x+4)(x+7)}$$

$$=\frac{5x+20}{(x+2)(x+7)(x+4)}+\frac{x^2+7x}{(x+2)(x+4)(x+7)}$$

$$=\frac{x^2+12x+20}{(x+2)(x+7)(x+4)}$$

$$=\frac{\cancel{(x+2)}(x+10)}{\cancel{(x+2)}(x+7)(x+4)}$$

$$=\frac{x+10}{(x+7)(x+4)} \qquad x\neq -2$$

Example 9 Simplify:

$$\frac{x}{x^2-x-6}-\frac{9}{x^2+9x-36}=\frac{x}{(x-3)(x+2)}-\frac{9}{(x+12)(x-3)}$$

$$=\frac{x(x+12)}{(x-3)(x+2)(x+12)}-\frac{9(x+2)}{(x+12)(x-3)(x+2)}$$

$$=\frac{x^2+12x}{(x-3)(x+2)(x+12)}-\frac{9x+18}{(x+12)(x-3)(x+2)}$$

$$=\frac{x^2+3x-18}{(x-3)(x+2)(x+12)}$$

$$=\frac{\cancel{(x-3)}(x+6)}{\cancel{(x-3)}(x+2)(x+12)}$$

$$=\frac{x+6}{(x+2)(x+12)} \qquad x\neq 3$$

6.3 Complex Fractions

We've already learned that a rational expression is a fraction with polynomials for the numerator and denominator.

If the numerator and denominator of a fraction are rational expressions themselves, the fraction is a complex fraction. These expressions are complicated, as their name suggests[1], so it is desirable to simplify them as much as possible.

If the numerator and denominator each contain only a single fraction, then the complex fraction is simply just division of two rational expressions, written in a different form. This means they can be treated in the exact same way, by multiplying by the reciprocal of the divisor.

Example 1 Simplify:

$$\frac{\frac{x+3}{x}}{\frac{x}{x+1}} = \frac{x+3}{x} \cdot \frac{x+1}{x}$$

$$= \frac{(x+3)(x+1)}{x^2}$$

If a complex fraction contains a sum or difference of rational expressions, then there are a couple of options to simplify them.

Method 1: Multiply by Denominators

In this method, we eliminate the denominators of the smaller fractions by multiplying everything by their factors.

Example 2 Simplify:

$$\frac{\frac{1}{x} + \frac{2}{x+5}}{\frac{x}{x+5}} = \frac{\frac{1}{x} \cdot x + \frac{2}{x+5} \cdot x}{\frac{x}{x+5} \cdot x} = \frac{1 + \frac{2x}{x+5}}{\frac{x^2}{x+5}}$$

$$= \frac{1 \cdot (x+5) + \frac{2x}{x+5} \cdot (x+5)}{\frac{x^2}{x+5} \cdot (x+5)} = \frac{x+5+2x}{x^2}$$

$$= \frac{3x+5}{x^2} \qquad x \neq -5$$

[1] The name "complex fractions" does not imply they are related to complex numbers. If you want a less confusing name, you could call them "nested fractions."

Method 2: Adding and Subtracting First

In this method, we simplify the numerator and/or the denominator as we would for any expression with addition or subtraction. Then treat the result as division.

Example 3 Simplify:

$$\frac{\frac{1}{x}+\frac{2}{x+5}}{\frac{x}{x+5}} = \frac{\frac{x+5}{x(x+5)}+\frac{2x}{(x+5)x}}{\frac{x}{x+5}} = \frac{\frac{3x+5}{x(x+5)}}{\frac{x}{x+5}}$$

$$= \frac{3x+5}{x\cancel{(x+5)}}\cdot\frac{\cancel{x+5}}{x}$$

$$= \frac{3x+5}{x^2} \qquad x \neq -5$$

Example 4 Simplify:

Using Method 1:

$$\frac{\frac{x-7}{x^2-9}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{x^2-9}} = \frac{\frac{x-7}{(x+3)(x-3)}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{(x+3)(x-3)}}$$

$$= \frac{\frac{x-7}{(x+3)(x-3)}\cdot(x+3)+\frac{2}{x+3}\cdot(x+3)}{\frac{5}{x-3}\cdot(x+3)-\frac{x+6}{(x+3)(x-3)}\cdot(x+3)} = \frac{\frac{x-7}{x-3}+2}{\frac{5x+15}{x-3}-\frac{x+6}{x-3}}$$

$$= \frac{\frac{x-7}{x-3}\cdot(x-3)+2\cdot(x-3)}{\frac{5x+15}{x-3}\cdot(x-3)-\frac{x+6}{x-3}\cdot(x-3)} = \frac{x-7+2x-6}{5x+15-x-6}$$

$$= \frac{3x-13}{4x+9} \qquad x \neq -3, 3$$

Using Method 2:

$$\frac{\frac{x-7}{x^2-9}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{x^2-9}} = \frac{\frac{x-7}{(x+3)(x-3)}+\frac{2}{x+3}}{\frac{5}{x-3}-\frac{x+6}{(x+3)(x-3)}}$$

$$= \frac{\frac{x-7}{(x+3)(x-3)}+\frac{2(x-3)}{(x+3)(x-3)}}{\frac{5(x+3)}{(x-3)(x+3)}-\frac{x+6}{(x+3)(x-3)}} = \frac{\frac{x-7}{(x+3)(x-3)}+\frac{2x-6}{(x+3)(x-3)}}{\frac{5x+15}{(x-3)(x+3)}-\frac{x+6}{(x+3)(x-3)}}$$

$$= \frac{\frac{3x-13}{(x+3)(x-3)}}{\frac{4x+9}{(x+3)(x-3)}} = \frac{3x-13}{4x+9} \qquad x \neq -3, 3$$

6.4 Rational Equations

An equation which consists of rational expressions is called a rational equation. As with any equation, solving means finding the values for the variable which make the equation true.

To simplify the equation, we can eliminate the denominators by multiplying the entire equation by their factors. This reduces the equation to a polynomial equation, which is frequently a quadratic equation. We can then use our typical methods to finish solving.

Example 1 Solve $\dfrac{x+2}{x-2} - \dfrac{x+9}{x} = 1$

$$\frac{x+2}{x-2}\cdot(x-2) - \frac{x+9}{x}\cdot(x-2) = 1\cdot(x-2)$$

$$x+2-\frac{x^2+7x-18}{x} = x-2$$

$$(x+2)\cdot x - \frac{x^2+7x-18}{x}\cdot x = (x-2)\cdot x$$

$$x^2+2x-x^2-7x+18 = x^2-2x$$

$$-x^2-3x+18=0$$

$$x^2+3x-18=0$$

$$(x+6)(x-3)=0$$

$$x=-6 \text{ or } x=3$$

We can check that both solutions are valid by substituting them into the original equation.

If $x=-6$, then LHS $= \frac{(-6)+2}{(-6)-2} - \frac{(-6)+9}{(-6)} = \frac{-4}{-8} - \frac{3}{-6} = 1$, RHS $=1$.

If $x=3$, then LHS $= \frac{(3)+2}{(3)-2} - \frac{(3)+9}{(3)} = \frac{5}{1} - \frac{12}{3} = 1$, RHS $=1$.

In this case, both of the solutions satisfy the equation. This is not always true, which is why we need to check the solutions.

For rational equations, it is possible to obtain extraneous solutions. Extraneous solutions, which are not actually solutions, appear when the equation is solved, but are inconsistent with the original equation.

Example 2 Solve $\dfrac{x-3}{x+3}+\dfrac{2}{x-2}=\dfrac{5x}{x^2+x-6}$

$$\frac{x-3}{x+3}\cdot(x+3)+\frac{2}{x-2}\cdot(x+3)=\frac{5x}{(x+3)(x-2)}\cdot(x+3)$$

$$x-3+\frac{2x+6}{x-2}=\frac{5x}{x-2}$$

$$(x-3)\cdot(x-2)+\frac{2x+6}{x-2}\cdot(x-2)=\frac{5x}{x-2}\cdot(x-2)$$

$$x^2-5x+6+2x+6=5x$$

$$x^2-8x+12=0$$

$$(x-2)(x-6)=0$$

$$\cancel{x=2} \text{ or } x=6$$

Checking the solutions:

If $x=2$, then LHS $=\frac{(2)-3}{(2)+3}+\frac{2}{(2)-2}$ is undefined, RHS $=\frac{5(2)}{(2)^2+(2)-6}$ is undefined.

If $x=6$, then LHS $=\frac{(6)-3}{(6)+3}+\frac{2}{(6)-2}=\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$, RHS $=\frac{5(6)}{(6)^2+(6)-6}=\frac{5}{6}$.

$\Longrightarrow$ $x=2$ is an extraneous solution, $x=6$ is the only solution.

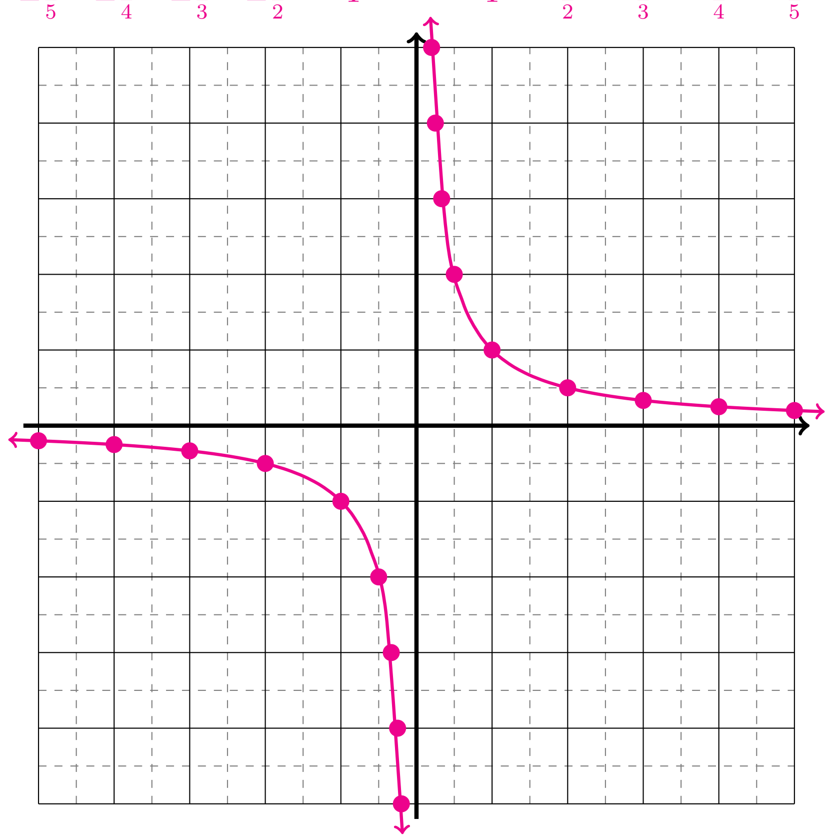
Because extraneous solutions can arise from rational equations, you must always check your solutions with the original equation.

6.5 Simple Rational Functions

The simplest non-trivial rational function is the reciprocal function.

parent function	$f(x) = x^{-1} = \dfrac{1}{x}$
domain	$\mathbb{R} \setminus \{0\}$
range	$\mathbb{R} \setminus \{0\}$
relation type	one-to-one
horizontal asymptote	$y = 0$
vertical asymptote	$x = 0$
shape	hyperbola

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	undef	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



An asymptote is a line which a function's curve continues to get closer to, without ever reaching it.

This function has a horizontal asymptote at $y = 0$, because as x increases towards $+\infty$ or decreases towards $-\infty$, $f(x)$ continues to get closer to zero.

$$\text{As } x \to \pm\infty,\ f(x) \to 0$$

The function also has a vertical asymptote at $x = 0$, because as x gets closer to zero, $f(x)$ continues to increase $+\infty$ or decrease to $-\infty$.

$$\text{As } x \to 0,\ f(x) \to \pm\infty$$

Transformations of the Reciprocal Function

By applying transformations to $y = \frac{1}{x}$, we arrive at the general form

$$f(x) = \frac{A}{x-h} + k$$

A sketch of this type of function should include:

shape of curve	hyperbola with enough points to show stretch/compression
x-intercept	$y=0$, find x by solving $f(x)=0$, exists if $k \neq 0$
y-intercept	$x=0$, find y by evaluating $y=f(0)$, exists if $h \neq 0$
vertical asymptote	$x=h$, as $f(h)$ is undefined
horizontal asymptote	$y=k$, as $f(x)=k$ has no solution
endpoints	evaluate the function at the bounds of the domain

The points one unit left and right of the vertical asymptote are useful for guiding the overall shape of the graph.

Example 1 Sketch a graph of $f(x) = \frac{-1}{x-3} - 5$, and state its domain and range in three forms.

Orientation: Inverted

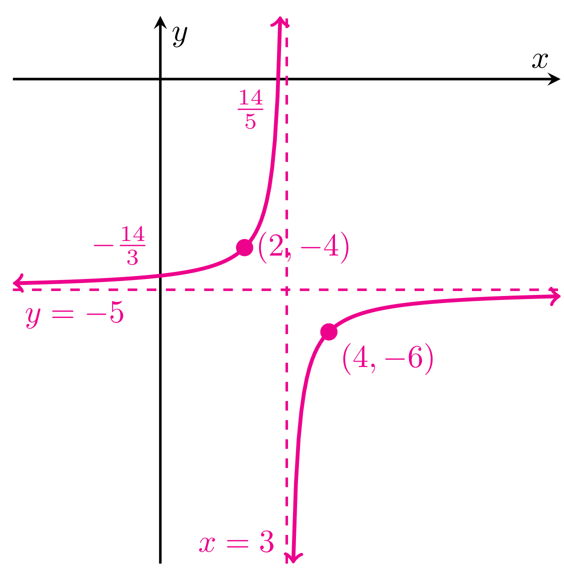
Asymptotes: $x = 3$ $y = -5$

x-intercept: $(\frac{14}{5}, 0)$

$$\frac{-1}{x-3} - 5 = 0$$
$$\frac{-1}{x-3} = 5$$
$$x - 3 = -\frac{1}{5}$$
$$x = 3 - \frac{1}{5} = \frac{14}{5}$$

y-intercept: $(0, -\frac{14}{3})$ as $f(0) = -\frac{14}{3}$

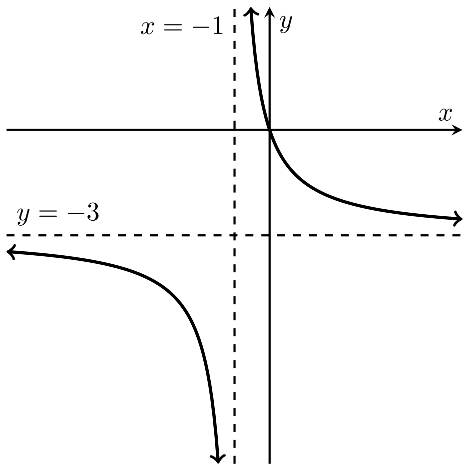
Other points: $f(2) = -4$ $f(4) = -6$



Domain:
$$\mathbb{R} \setminus \{3\}$$
$$= (-\infty, 3) \cup (3, \infty)$$
$$= \{x : x \neq 3\}$$

Range:
$$\mathbb{R} \setminus \{-5\}$$
$$= (-\infty, -5) \cup (-5, \infty)$$
$$= \{y : y \neq -5\}$$

Example 2 Find the function g represented by the following graph.



asymptotes: $x=-1$ and $y=-3$

$\implies h=-1, k=-3$

$g(x)=\frac{A}{x+1}-3$

passes through origin: $g(0)=0$

$\frac{A}{1}-3=0 \implies A=3$

$$g(x)=\frac{3}{x+1}-3$$

Inverses of Simple Rational Functions

Functions of the form $y=\frac{A}{x-h}+k$ are one-to-one, which means they each have an inverse function. It turns out that the inverse functions have the same form. Finding inverses follows the same process we used in section 2.2.

Example 3 Find the inverse of $f(x)=\dfrac{1}{x-2}+7$. State the domain and range of f, and the domain and range of f^{-1}.

$$y=\frac{1}{x-2}+7$$

swap $x \leftrightarrow y$: $$x=\frac{1}{y-2}+7$$

$$x-7=\frac{1}{y-2}$$

$$(x-7)(y-2)=1$$

$$y-2=\frac{1}{x-7}$$

$$y=\frac{1}{x-7}+2$$

$$f^{-1}(x)=\frac{1}{x-7}+2$$

domain of $f=\mathbb{R}\setminus\{2\}$

range of $f=\mathbb{R}\setminus\{7\}$

domain of $f^{-1}=\mathbb{R}\setminus\{7\}$

range of $f^{-1}=\mathbb{R}\setminus\{2\}$

Linear Rational Functions

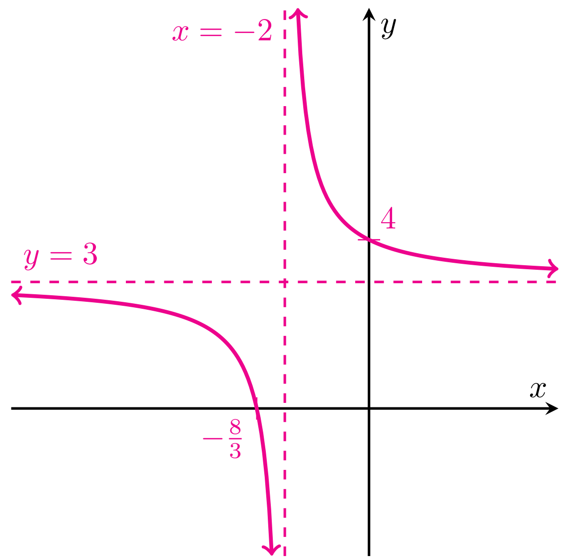
A rational function whose numerator and denominator are both _linear_ has a _hyperbola_ for its graph, just like $y = \frac{A}{x-h} + k$, though determining its characteristics is more difficult. To handle these functions, we can use _polynomial division_ (section 5.4) to convert their form.

Example 4 Write $f(x) = \dfrac{3x+8}{x+2}$ in the form $y = \frac{A}{x-h} + k$, and sketch its graph.

You can use the known values $f(0) = 4$ and $f(-\frac{8}{3}) = 0$.

	3	R
x	$3x$	$+2$
$+2$	$+6$	

$$f(x) = \frac{3x+8}{x+2} = 3 + \frac{2}{x+2}$$
$$= \frac{2}{x+2} + 3$$



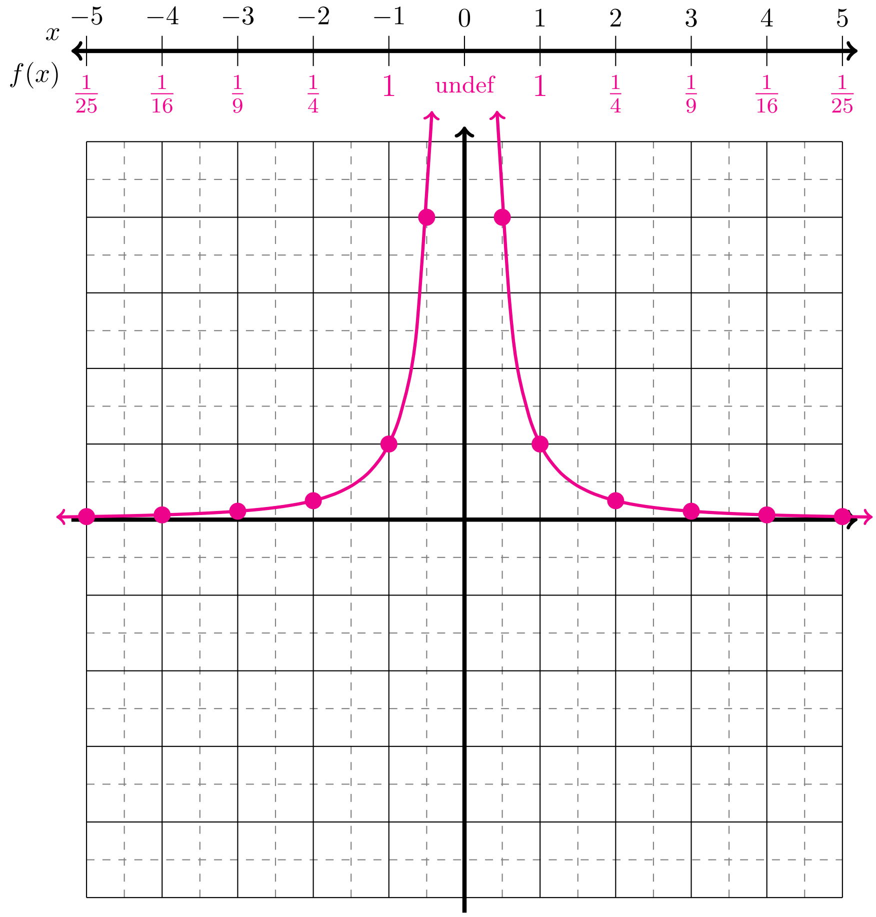
Example 5 Write $g(x) = \dfrac{-2}{x-6} + 7$ in the form $y = \dfrac{ax+b}{cx+d}$.

$$\begin{aligned} g(x) &= \frac{-2}{x-6} + 7 \\ &= \frac{-2}{x-6} + \frac{7(x-6)}{x-6} \\ &= \frac{-2+7x-42}{x-6} \\ &= \frac{7x-44}{x-6} \end{aligned}$$

6.6 Functions with Quadratic Denominators

Transformations of x^{-2}

parent function $f(x) = x^{-2} = \dfrac{1}{x^2}$
domain $\mathbb{R} \setminus \{0\}$
range $(0, \infty)$
relation type many-to-one
horizontal asymptote $y = 0$
vertical asymptote $x = 0$
shape truncus



This parent function is similar to the reciprocal function. It has the same domain, and its graph has the same asymptotes. However, because x is squared, the output values are all positive, which changes the range.

Note that the shape of a curve is not a hyperbola, but is a slightly different shape called a truncus.

By applying transformations, we arrive at the general form

$$f(x) = \frac{A}{(x-h)^2} + k$$

A sketch of this type of function should include:

shape of curve	truncus with enough points to show stretch/compression
x-intercepts	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$, exists if $h \neq 0$
vertical asymptote	$x = h$, as $f(h)$ is undefined
horizontal asymptote	$y = k$, as $f(x) = k$ has no solution
endpoints	evaluate the function at the bounds of the domain

Example 1 Sketch a graph of $f(x) = \dfrac{9}{(x-7)^2} - 4$.

Asymptotes: $x = 7$ $\quad y = -4$

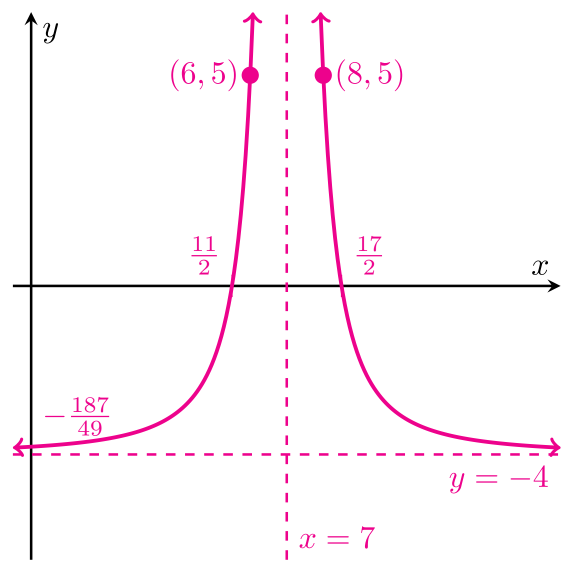
x-intercept: $(\frac{11}{2}, 0)$ and $(\frac{17}{2}, 0)$

$$\begin{aligned} \tfrac{9}{(x-7)^2} - 4 &= 0 \\ \tfrac{9}{(x-7)^2} &= 4 \\ (x-7)^2 &= \tfrac{9}{4} \\ x - 7 &= \pm\tfrac{3}{2} \\ x &= 7 \pm \tfrac{3}{2} \end{aligned}$$

y-intercept: $(0, -\frac{187}{49})$

as $f(0) = -\frac{187}{49} \approx -3.816$

Other points: $f(6) = 5$ $\quad f(8) = 5$



Example 2 Find the rule for a rational function f with an implied domain of $(-\infty, -2) \cup (-2, \infty)$ and a range of $(-\infty, 8)$. The function does not represent a stretch or compression applied to the parent function.

No stretch or compression $\implies A = \pm 1$.

$f(x) < 8 \implies$ graph is inverted $\implies$ A is negative $\implies A = -1$

Asymptotes are $x = -2$ and $y = 8 \implies h = -2$ and $k = 8$

$$f(x) = \frac{-1}{(x+2)^2} + 8$$

Reciprocals of Quadratic Functions

Functions of the form $f(x) = \frac{1}{q(x)}$, where $q(x)$ is a quadratic function, can be graphed by examining the behavior of $q(x)$.

If quadratic function $q(x)$...	...then its reciprocal $f(x) = \frac{1}{q(x)}$...
has a zero at x	has a vertical asymptote at x
has a local minimum (h, k)	has a local maximum $(h, \frac{1}{k})$
has a local maximum (h, k)	has a local minimum $(h, \frac{1}{k})$
approaches $\pm\infty$	approaches zero (asymptote $y = 0$)
is positive	is positive
is negative	is negative
equals ± 1	equals ± 1

Example 3 Draw the graph of $f(x) = \frac{1}{x^2 - 8x + 12}$. The graph of $q(x) = x^2 - 8x + 12$ is already shown.

Asymptotes: $y = 0, x = 2, x = 6$

Vertical asymptotes when $q(x) = 0$:

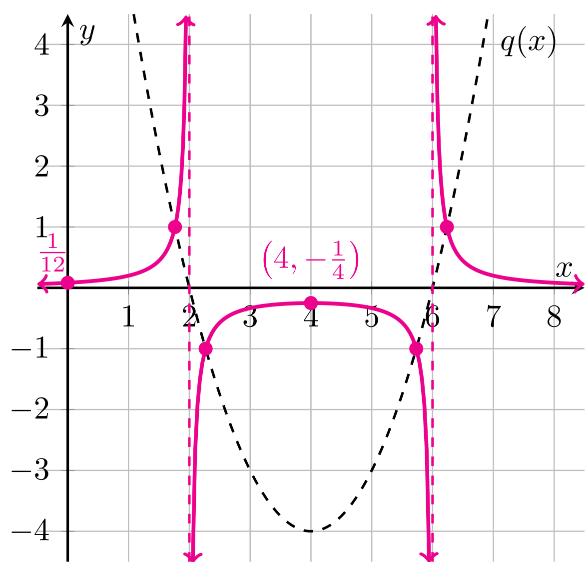
$$x^2 - 8x + 12 = 0$$
$$(x-2)(x-6) = 0$$
$$x = 2 \text{ or } x = 6$$

y-intercept: $(0, \frac{1}{12})$ as $f(0) = \frac{1}{12}$

Vertex: $(4, -\frac{1}{4})$

$q(x)$ has vertex at $(4, -4)$ because $\frac{2+6}{2} = 4$,
$q(4) = (4)^2 - 8(4) + 12 = -4$

Points where $f(x) = q(x) = \pm 1$ are marked.



Note that you won't typically be given the parabola for the quadratic in practice questions. It's still a good idea to draw it first before attempting to draw its reciprocal.

Example 4 Sketch a graph of $f(x) = \dfrac{2}{x^2 - 4x + 5}$.

Rewrite $f(x)$ in the form $\dfrac{1}{q(x)}$: $f(x) = \dfrac{1}{\frac{1}{2}x^2 - 2x + \frac{5}{2}}$

Properties of $q(x)$:

Zeros: none

y-intercept: $\left(0, \frac{5}{2}\right)$

Vertex: $\left(2, \frac{1}{2}\right)$

Equals ± 1: $(1, 1)$ and $(3, 1)$

Properties of $f(x)$:

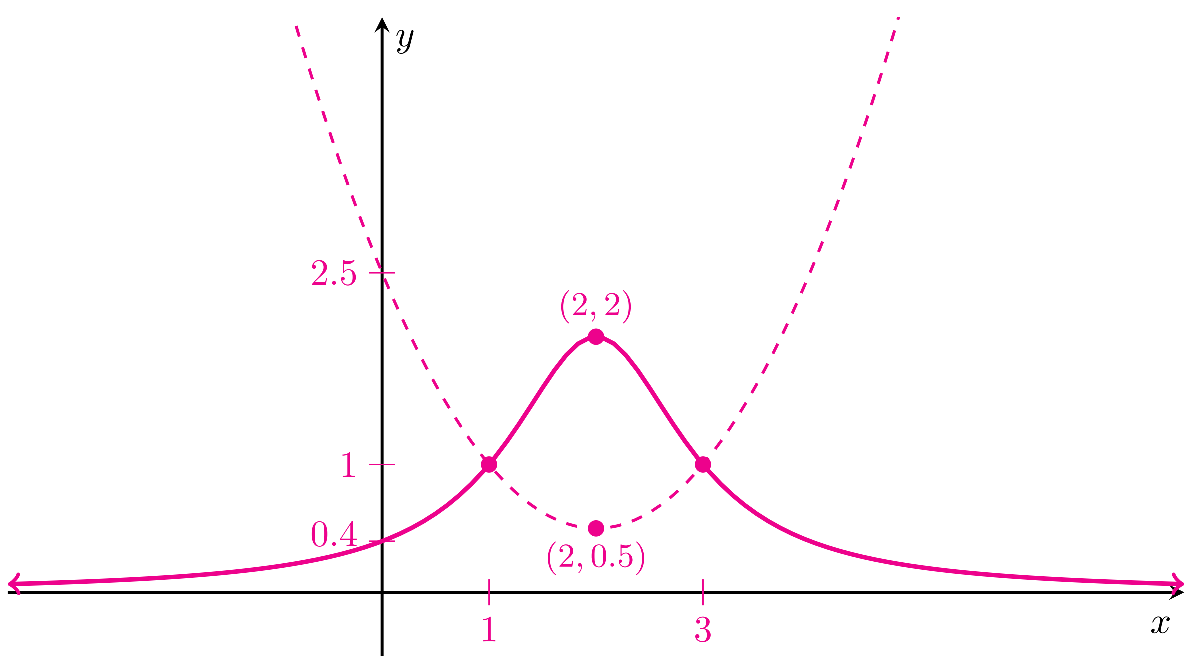
Vertical Asymptotes: none

y-intercept: $\left(\frac{5}{2}\right)^{-1} = \frac{2}{5} \implies \left(0, \frac{2}{5}\right)$

Vertex: $\left(\frac{1}{2}\right)^{-1} = 2 \implies (2, 2)$

Equals ± 1: $(1, 1)$ and $(3, 1)$

Horizontal Asymptote: $y = 0$



Chapter 7

Radicals and Rational Exponents

7.1 Radical Expression Concepts . . . 118
7.2 Rational Exponents . . . 121
7.3 Square Root Equations . . . 124
7.4 Square Root Functions . . . 126
7.5 Cube Root Functions . . . 130
7.6 Quadratics, Cubics and Roots as Inverses . . . 132

7.1 Radical Expression Concepts

Recall that the nth root of x is the value y such that $y^n = x$, which we write as

$$y = \sqrt[n]{x}$$

- The symbol $\sqrt{}$ is the radical symbol.
- The small number written over the radical n is called the index. (Don't mix this up with a **coefficient** written in front of the radical.)
- The value x under the radical is called the radicand.

The 2nd root is called the square root, and is usually written without the index.

The 3rd root is called the cube root.

Example 1

$\sqrt{81} = 9$ because $9^2 = 81$

$\sqrt[3]{125} = 5$ because $5^3 = 125$

$\sqrt[5]{32} = 2$ because $2^5 = 32$

Simplifying Radicals

It is conventional to write radical expressions with the smallest possible value in the radicand. This is done by identifying a factor which has a rational nth root.

Example 2 Simplify the following.

$$\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

$$\sqrt[3]{108} = \sqrt[3]{27}\sqrt[3]{4} = 3\sqrt[3]{4}$$

$$\sqrt[6]{128} = \sqrt[6]{64}\sqrt[6]{2} = 2\sqrt[6]{2}$$

The same principle can be used when there are variables in the radicand.

Example 3 Simplify the following.

$$\sqrt{75x^7} = \sqrt{25x^6}\sqrt{3x} = 5x^3\sqrt{3x}$$

$$\sqrt[3]{48x^5} = \sqrt[3]{8x^3}\sqrt[3]{6x^2} = 2x\sqrt[3]{6x^2}$$

$$\sqrt[4]{81xy^5} = \sqrt[4]{81y^4}\sqrt[4]{xy} = 3y\sqrt[4]{xy}$$

Adding and Subtracting Radicals

Radical terms with the same radicand and index can be added or subtracted by adding or subtracting their coefficients, just as like terms are simplified.

Some radicals may need to be simplified first.

Example 4 Simplify the following.

$$9\sqrt{6} - 7\sqrt{3} + \sqrt{6} + 4\sqrt{3} = 10\sqrt{6} - 3\sqrt{3}$$

Example 5 Simplify the following.

$$\begin{aligned} 2\sqrt{45} + 3\sqrt{50} - 6\sqrt{8} + 4\sqrt{20} &= 2\sqrt{9}\sqrt{5} + 3\sqrt{25}\sqrt{2} - 6\sqrt{4}\sqrt{2} + 4\sqrt{4}\sqrt{5} \\ &= 2 \cdot 3\sqrt{5} + 3 \cdot 5\sqrt{2} - 6 \cdot 2\sqrt{2} + 4 \cdot 2\sqrt{5} \\ &= 6\sqrt{5} + 15\sqrt{2} - 12\sqrt{2} + 8\sqrt{5} \\ &= 14\sqrt{5} + 3\sqrt{2} \end{aligned}$$

Multiplying Radicals

Radicals with the same index can be multiplied by multiplying their radicands. If each radical has a coefficient, these are multiplied together.

Example 6 Simplify the following.

$$\begin{aligned} 3\sqrt{10} \cdot 7\sqrt{2} &= 21\sqrt{20} \\ &= 21 \cdot 2\sqrt{5} \\ &= 42\sqrt{5} \end{aligned}$$

$$\begin{aligned} 2\sqrt{7} \cdot 5\sqrt{14} &= 10\sqrt{98} \\ &= 10 \cdot 7\sqrt{2} \\ &= 70\sqrt{2} \end{aligned}$$

If binomial expressions are being multiplied, then we can use the distributive property.

Example 7 Simplify the following.

$$\begin{aligned} 3\sqrt{2}(\sqrt{5} + 4\sqrt{2}) &= 3\sqrt{10} + 12\sqrt{4} \\ &= 3\sqrt{10} + 24 \end{aligned}$$

	$\sqrt{5}$	$+4\sqrt{2}$
$3\sqrt{2}$	$3\sqrt{10}$	$+24$

Example 8 Simplify the following.

$$\begin{aligned} (2 + \sqrt{5})(7 - 6\sqrt{5}) &= 14 - 12\sqrt{5} + 7\sqrt{5} - 6\sqrt{25} \\ &= 14 - 12\sqrt{5} + 7\sqrt{5} - 30 \\ &= -16 - 5\sqrt{5} \end{aligned}$$

	2	$+\sqrt{5}$
7	14	$+7\sqrt{5}$
$-6\sqrt{5}$	$-12\sqrt{5}$	-30

Dividing Radicals

When dividing radicals, it is considered good practice to ensure the denominator is rational, in a process called rationalizing the denominator.

If the denominator has one term, we can multiply by an appropriate radical to make it rational. In the case of a square root, we can use the same square root.

Example 9 Rationalize the denominators.

$$\frac{3\sqrt{7}}{5\sqrt{3}} = \frac{3\sqrt{7}}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{21}}{15} = \frac{\sqrt{21}}{5}$$

$$\frac{4\sqrt[3]{6}}{3\sqrt[3]{2}} = \frac{4\sqrt[3]{6}}{3\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{4\sqrt[3]{24}}{3\sqrt[3]{8}} = \frac{8\sqrt[3]{3}}{6} = \frac{4\sqrt[3]{3}}{3}$$

If the denominator has two terms involving square roots (but not higher roots), we can make it rational by multiplying by its conjugate, following the same process we used for dividing complex numbers in section 4.1.

Example 10 Rationalize the denominator.

$$\frac{6\sqrt{2}+7\sqrt{3}}{3\sqrt{2}+5\sqrt{3}} = \frac{6\sqrt{2}+7\sqrt{3}}{3\sqrt{2}+5\sqrt{3}} \cdot \frac{3\sqrt{2}-5\sqrt{3}}{3\sqrt{2}-5\sqrt{3}}$$

$$= \frac{18\sqrt{4}+21\sqrt{6}-30\sqrt{6}-35\sqrt{9}}{9\sqrt{4}+15\sqrt{6}-15\sqrt{6}-25\sqrt{9}}$$

$$= \frac{-69-9\sqrt{6}}{-57}$$

$$= \frac{69+9\sqrt{6}}{57}$$

7.2 Rational Exponents

Review of Exponents

An exponent is used to indicate repeated multiplication of a number called the base.

$$a^n = \underbrace{a \cdot a \cdot a \ldots a}_{n \text{ times}}$$

where n is the exponent and a is the base.

Exponent Product Rule

$$a^m \cdot a^n = a^{m+n}$$

Exponent Quotient Rule

$$\frac{a^m}{a^n} = a^{m-n}$$

Exponent Power Rule

$$(a^m)^n = a^{mn}$$

Negative Exponent Rule

$$a^{-n} = \frac{1}{a^n}$$

Base Product Rule

$$(ab)^n = a^n b^n$$

Base Quotient Rule

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Special Value Zero

$$a^0 = 1 \quad (a \neq 0)$$

Special Value One

$$a^1 = a$$

Rational Exponents

When an exponent is a fraction, it is known as a rational exponent. We can use the Exponent Power Rule to help evaluate them.

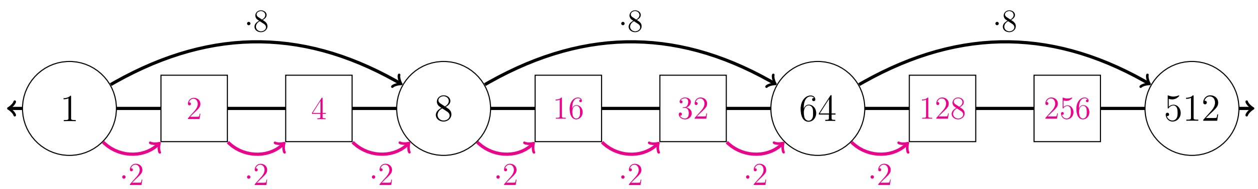
Example 1 Evaluate the following.

$$36^{1/2} = \left(6^2\right)^{1/2} = 6^{2 \cdot 1/2} = 6^1 = 6$$

$$81^{3/4} = \left(3^4\right)^{3/4} = 3^{4 \cdot 3/4} = 3^3 = 27$$

$$8^{7/3} = \left(2^3\right)^{7/3} = 2^{3 \cdot 7/3} = 2^7 = 128$$

Let's take a closer look at the last example and consider what $8^{7/3}$ actually means. Recall that an exponent indicates how many times the base is multiplied by itself. From the diagram it's simple to see that, for instance, multiplying by 8 three times results in $8^3 = 512.$



But what does it mean to multiply 8 seven-thirds times, since it is not an integer? Consider that multiplying by 8 once is the same as multiplying by 2 three times. It follows that multiplying by 8 "one-third times" is equivalent to multiplying by 2 once.

Finally, this means that multiplying by 8 seven-thirds times is the same as multiplying by 2 seven times, and that $8^{7/3} = 128.$

Roots and Exponents

Consider the following:

$$\sqrt{36} = \sqrt{6^2} = 6$$

$$\left(\sqrt[4]{81}\right)^3 = \left(\sqrt[4]{3^4}\right)^3 = 3^3 = 27$$

$$\left(\sqrt[3]{8}\right)^7 = \left(\sqrt[3]{2^3}\right)^7 = 2^7 = 128$$

Notice that we're performing the same calculations as the example above, with the index of the root taking the place of the denominator of the exponent. This is because radicals and rational exponents are equivalent.

Theorem: Roots and Rational Exponents

$$\sqrt[n]{x} = x^{1/n} \qquad \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m = x^{m/n}$$

Proof

$$\begin{aligned}
\text{Let } y &= \sqrt[n]{x} \\
\implies x &= y^n && \text{definition of the nth root} \\
x^{1/n} &= (y^n)^{1/n} \\
&= y^{n \cdot 1/n} && \text{exponent power rule} \\
&= y \\
\implies \sqrt[n]{x} &= x^{1/n} \\
\sqrt[n]{x^m} &= (x^m)^{1/n} = x^{m/n} \\
\left(\sqrt[n]{x}\right)^m &= \left(x^{1/n}\right)^m = x^{m/n}
\end{aligned}$$

■

Example 2 Write the following in exponent form.

$\sqrt[5]{11} = 11^{1/5}$ $\sqrt{6^9} = 6^{9/2}$ $(\sqrt[4]{21})^{13} = 21^{13/4}$

Example 3 Write the following in radical form.

$7^{1/6} = \sqrt[6]{7}$ $31^{5/3} = \sqrt[3]{31^5}$ $10^{11/2} = \sqrt{10^{11}}$

Example 4 Evaluate the following.

$$25^{1/2} = \sqrt{25} = 5$$

$$32^{3/5} = (\sqrt[5]{32})^3 = 2^3 = 8$$

$$343^{4/3} = (\sqrt[3]{343})^4 = 7^4 = 2401$$

Example 5 Simplify the following.

$$\left(\sqrt[4]{x}\right)^{12} = x^{12/4} = x^3$$

$$\sqrt[6]{x^3} = x^{3/6} = x^{1/2} = \sqrt{x}$$

$$\sqrt[12]{16} = \left(2^4\right)^{1/12} = 2^{4/12} = 2^{1/3} = \sqrt[3]{2}$$

7.3 Square Root Equations

Recall that to solve rational equations, we converted them into polynomial equations, which we then solved using the usual methods. For equations with square roots we can take a similar approach.

Like rational equations, equations with square roots can have extraneous solutions, so each solution needs to be checked against the original equation.

Example 1 Solve $x = \sqrt{7x+15} - 1$.

Step 1: Rearrange the equation to isolate the square root.	$x = \sqrt{7x+15} - 1$ $x + 1 = \sqrt{7x+15}$
Step 2: Eliminate the square root by squaring both sides.	$(x+1)^2 = (\sqrt{7x+15})^2$ $x^2 + 2x + 1 = 7x + 15$
Step 3: Solve the resulting equation.	$x^2 - 5x - 14 = 0$ $(x-7)(x+2) = 0$ $x = 7$ or $x = -2$
Step 4: Check for extraneous solutions.	If $x = 7$: LHS $= 7$ RHS $= \sqrt{7(7)+15} - 1$ $= \sqrt{64} - 1$ $= 8 - 1$ $= 7$ (valid) If $x = -2$: LHS $= -2$ RHS $= \sqrt{7(-2)+15} - 1$ $= \sqrt{1} - 1$ $= 1 - 1$ $= 0$ (extraneous)
Step 5: State the valid solutions.	$\Longrightarrow x = 7$

Equations with multiple square roots are more challenging to solve, and require squaring more than once, as only one root can by isolated at a time. Care is needed to apply the perfect square rule appropriately.

Example 2 Solve $\sqrt{x+4}+3=\sqrt{7x+1}$.

$$\begin{aligned}
\sqrt{x+4}+3 &= \sqrt{7x+1} \\
\left(\sqrt{x+4}+3\right)^2 &= \left(\sqrt{7x+1}\right)^2 \\
(\sqrt{x+4})^2+2\cdot\sqrt{x+4}\cdot 3+3^2 &= 7x+1 \\
x+4+6\sqrt{x+4}+9 &= 7x+1 \\
6\sqrt{x+4} &= 6x-12 \\
\sqrt{x+4} &= x-2 \\
\left(\sqrt{x+4}\right)^2 &= (x-2)^2 \\
x+4 &= x^2-4x+4 \\
x^2-5x &= 0 \\
x(x-5) &= 0 \\
x=0 &\text{ or } x=5
\end{aligned}$$

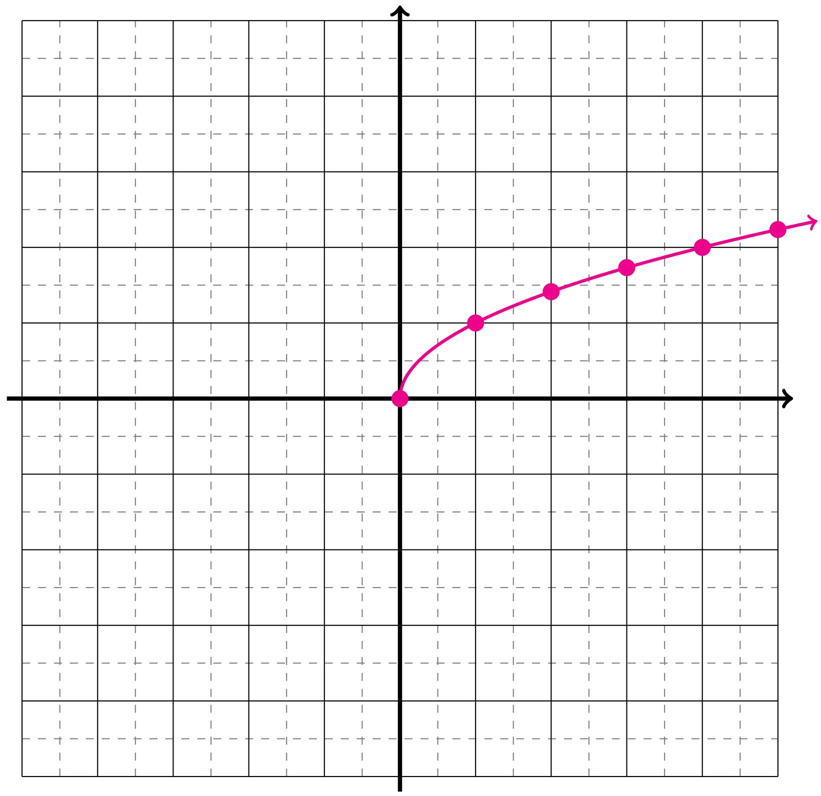
$$\begin{aligned}
&\text{If } x=0: \quad \text{LHS} = \sqrt{4}+3=5 \\
&\qquad\qquad\;\; \text{RHS} = \sqrt{1}=1 \qquad \text{(extraneous)} \\
&\text{If } x=5: \quad \text{LHS} = \sqrt{5+4}+3=6 \\
&\qquad\qquad\;\; \text{RHS} = \sqrt{7(5)+1}=6 \qquad \text{(valid)} \\
&\Longrightarrow x=5
\end{aligned}$$

7.4 Square Root Functions

Functions which contain a _radical_ can be called _radical functions_. For this class, we will consider _square root_ and _cube root_ functions.[1]

parent function	$f(x) = \sqrt{x} = x^{1/2}$
domain	$[0, \infty)$
range	$[0, \infty)$
relation type	one-to-one
x-intercept	$(0, 0)$
y-intercept	$(0, 0)$
endpoint	$(0, 0)$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$					← undefined	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$



As the inverse of _quadratic_ functions, square root functions have _parabolas_ for their curves, though facing a different direction. Half of the _parabola_ is missing; if the bottom half was present, it would not be a _function_.

Because the square root is _undefined_ for _negative_ numbers, all the _negative_ real numbers are excluded from the _implied domain_ of the parent function. We need to make sure that all square roots have only _positive_ numbers or _zero_ under them.

[1]We also only consider real-valued functions in this class. So, even though we know that $\sqrt{-1} = i$, for instance, we'll treat is as undefined in this section.

Example 1 Find the domain and range of $f(x) = -2\sqrt{x+4} + 6$.

$$x + 4 \geq 0$$
$$x \geq -4$$
$$\text{domain} = [-4, \infty)$$

$$\sqrt{x+4} \geq 0$$
$$-2\sqrt{x+4} \leq 0$$
$$f(x) = -2\sqrt{x+4} + 6 \leq 6$$
$$\text{range} = (-\infty, 6]$$

Example 2 Find the domain and range of $g(x) = \sqrt{-6(x-2)} + 5$.

$$-6(x-2) \geq 0$$
$$x - 2 \leq 0$$
$$x \leq 2$$
$$\text{domain} = (-\infty, 2]$$

$$\sqrt{-6(x-2)} \geq 0$$
$$g(x) = \sqrt{-6(x-2)} + 5 \geq 5$$
$$\text{range} = [5, \infty)$$

By applying transformations to the parent function, we get the general form of the square root function:

$$f(x) = A\sqrt{n(x-h)} + k$$

Recall from section 1.4 that n represents

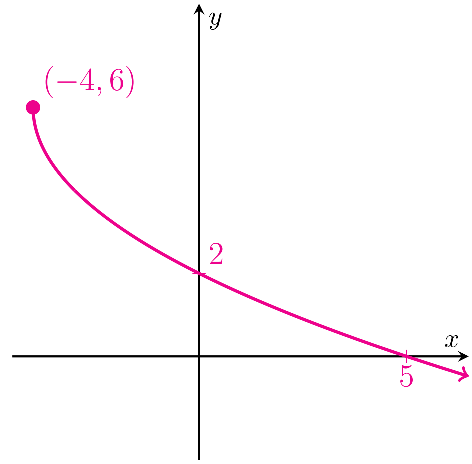
- a reflection across the y-axis if n is negative
- a stretch from the y-axis by a factor of $\frac{1}{|n|}$ if $0 < |n| < 1$
- a compression toward the y-axis by a factor of $|n|$ if $|n| > 1$

For our previous parent functions, their symmetry meant that all reflections could be represented with only A. This function has no symmetry, so n is needed as well.

A sketch of a square root function should include:

shape of curve	"half" parabola with enough points to show stretch/compression
x-intercept	$y = 0$, find x by solving $f(x) = 0$, may not exist
y-intercept	$x = 0$, find y by evaluating $y = f(0)$, may not exist
endpoint	(h, k), using translation of parent function to identify may be different with restricted domain

Example 3 Sketch a graph of $f(x) = -2\sqrt{x+4}+6$.



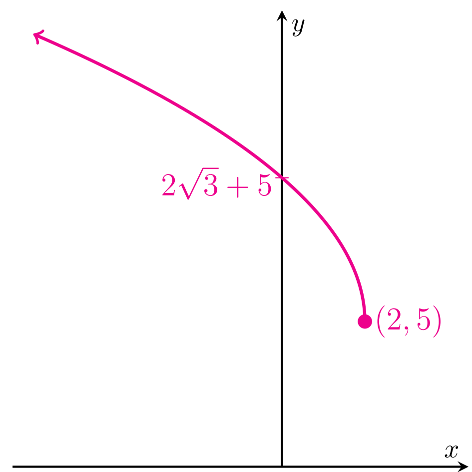
x-intercept: $(5, 0)$

$$-2\sqrt{x+4}+6=0$$
$$-2\sqrt{x+4}=-6$$
$$\sqrt{x+4}=3$$
$$x+4=9$$
$$x=5$$

y-intercept: $(0, 2)$ as $f(0) = -2\sqrt{4}+6 = 2$

endpoint: $(-4, 6)$

Example 4 Sketch a graph of $g(x) = \sqrt{-6(x-2)}+5$.



x-intercept: none

$$\sqrt{-6(x-2)}+5=0$$
$$\sqrt{-6(x-2)}=-5$$

No solution as square root can't be negative.

y-intercept: $(0, 2\sqrt{3}+5)$

$$g(0) = \sqrt{-6(-2)}+5 = \sqrt{12}+5$$
$$= 2\sqrt{3}+5 \approx 8.464$$

endpoint: $(2, 5)$

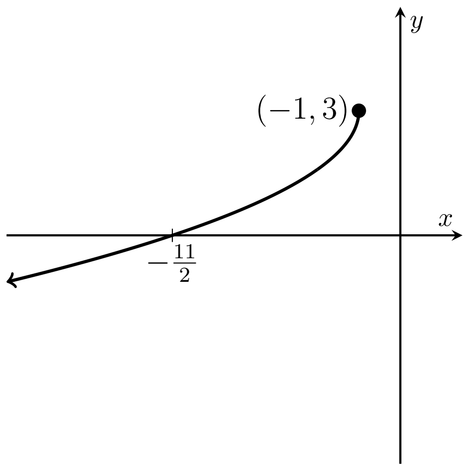
Example 5 List the transformations required to transform $f(x) = x^{1/2}$ to $g(x) = (-2x+5)^{1/2} - 3$.

To identify the transformations, we need to factor the inner part of g:

$$g(x) = \left[-2\left(x-\tfrac{5}{2}\right)\right]^{1/2} - 3$$

- Reflect across the y-axis.
- Compress towards the y-axis by a factor by a factor of 2.
- Shift $\frac{5}{2}$ units right.
- Shift 3 units down.

Example 6 Find the function f represented by the following graph.



endpoint: $(-1,3) \implies h=-1, k=3$

Graph is reflected across the x-axis.

$$\implies f(x) = -\sqrt{n(x+1)}+3$$

$$\begin{aligned} f(-\frac{11}{2}) &= -\sqrt{n(-\frac{11}{2}+1)}+3 \\ &= 0 \\ \sqrt{-\frac{9}{2}n} &= 3 \\ -\frac{9}{2}n &= 9 \\ n &= -2 \end{aligned}$$

$$f(x) = -\sqrt{-2(x+1)}+3$$

Example 7 The parent function $f(x)=\sqrt{x}$ is compressed toward the x-axis by a factor of 5. What horizontal transformation results in the same function?

Let g be the resulting function.

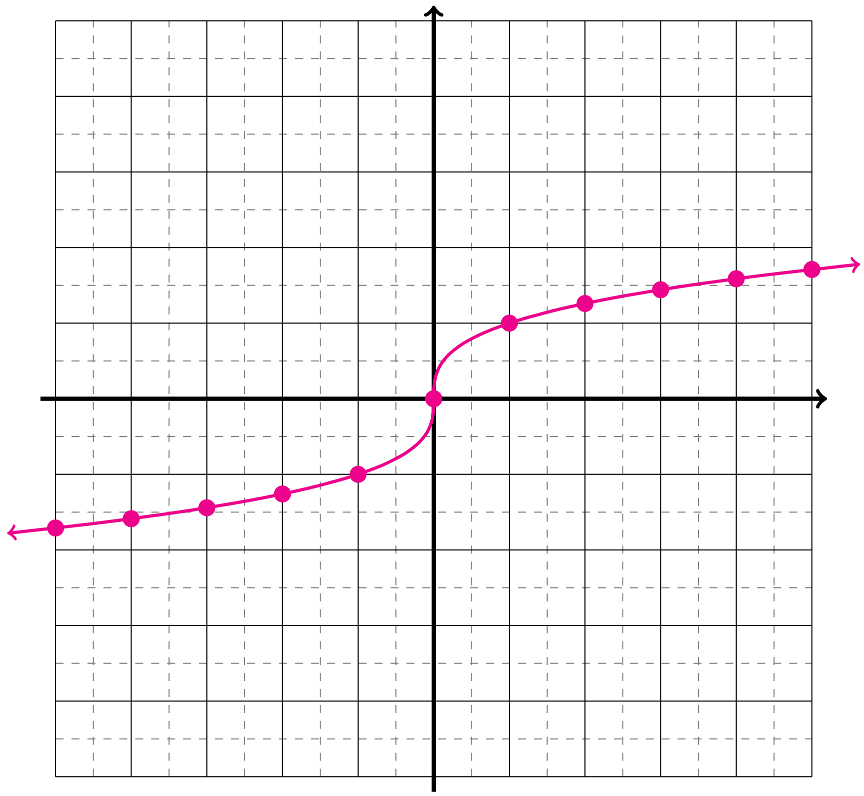
$$\begin{aligned} g(x) &= \tfrac{1}{5}\sqrt{x} \\ &= \sqrt{\tfrac{1}{25}}\sqrt{x} \\ &= \sqrt{\tfrac{1}{25}x} \end{aligned}$$

which corresponds to a stretch from the y-axis by a factor of 25.

7.5 Cube Root Functions

parent function	$f(x) = \sqrt[3]{x} = x^{1/3}$
domain	$\mathbb{R}$
range	$\mathbb{R}$
relation type	one-to-one
x-intercept	$(0, 0)$
y-intercept	$(0, 0)$
point of inflection[2]	$(0, 0)$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	$-\sqrt[3]{5}$	$-\sqrt[3]{4}$	$-\sqrt[3]{3}$	$-\sqrt[3]{2}$	-1	0	1	$\sqrt[3]{2}$	$\sqrt[3]{3}$	$\sqrt[3]{4}$	$\sqrt[3]{5}$



Unlike the square root, the cube root can be evaluated for negative real numbers, which simplifies finding the domain and range for cube root functions, which are both all real numbers if there is no domain restriction.

As the inverse of the cubic parent function, $y = x^3$, the curve of the cube root function has the same shape, reflected over the line $y = x$.

Using transformations, we can write the general form for a cube root function

$$f(x) = A\sqrt[3]{x - h} + k$$

[2]This point does fit the definition of inflection we've used, because the curve changes from concave up to concave down here, but there are other ways to define inflection which would technically exclude this point. The distinction doesn't matter in this class, but does in Calculus. Alternatively, this could be called a *vertical tangent point*.

Example 1 Sketch a graph of $f(x) = -3\,(x-8)^{1/3}+6$.

point of inflection: $(8, 6)$

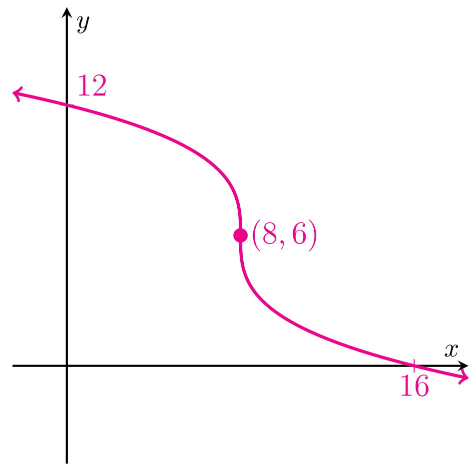
x-intercept: $(16, 0)$

$$-3\,(x-8)^{1/3}+6=0$$
$$-3\,(x-8)^{1/3}=-6$$
$$(x-8)^{1/3}=2$$
$$x-8=8$$
$$x=16$$

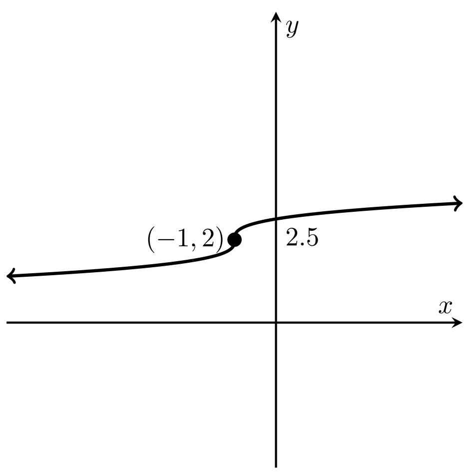
y-intercept: $(0, 12)$

as $f(0) = -3(-8)^{1/3}+6=12$

endpoints: none, as domain is $\mathbb{R}$



Example 2 Find the function g represented by the following graph.



Point of inflection: $(-1, 2) \implies h=-1, k=2$

$g(x) = A\sqrt[3]{x+1}+2$

$g(0) = \frac{5}{2}$

$A\sqrt[3]{1}+2=\frac{5}{2} \implies A+2=\frac{5}{2} \implies A=\frac{1}{2}$

$$g(x)=\tfrac{1}{2}\sqrt[3]{x+1}+2$$

7.6 Quadratics, Cubics and Roots as Inverses

Recall the following theorem:

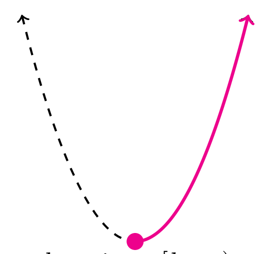
Theorem

A function f has an inverse function f^{-1} if and only if f is a one-to-one function.

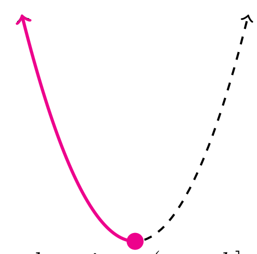
A cubic function of the form $f(x) = A(x-h)^3 + k$ is one-to-one, so it will always have an inverse function. The inverse will be a cube root function.

A quadratic function is more challenging because it is many-to-one, so does not have an inverse function. To get around this problem, we can restrict the domain of the function.

The resulting inverse will be a square root function.



domain $= [h, \infty)$



domain $= (-\infty, h]$

Theorem

Suppose f is a quadratic function, and that $y = f(x)$ has a vertex at (h, k).

If the domain of f is $[h, \infty)$ or $(-\infty, h]$, then f is one-to-one.

It is easiest to find the inverse of a quadratic functions in vertex form.

Example 1 Consider the function $f : [2, \infty) \to \mathbb{R}$, where $f(x) = (x-2)^2 - 4$.

a) Show that the inverse function f^{-1} exists.

$h = 2$

Domain is $[2, \infty) \implies f$ is one-to-one $\implies f^{-1}$ exists.

b) Find the range of f, and hence, the domain of f^{-1}.

$y = f(x)$ is upright $\implies k = -4$ is a minimum.

range of f = domain of $f^{-1} = [-4, \infty)$

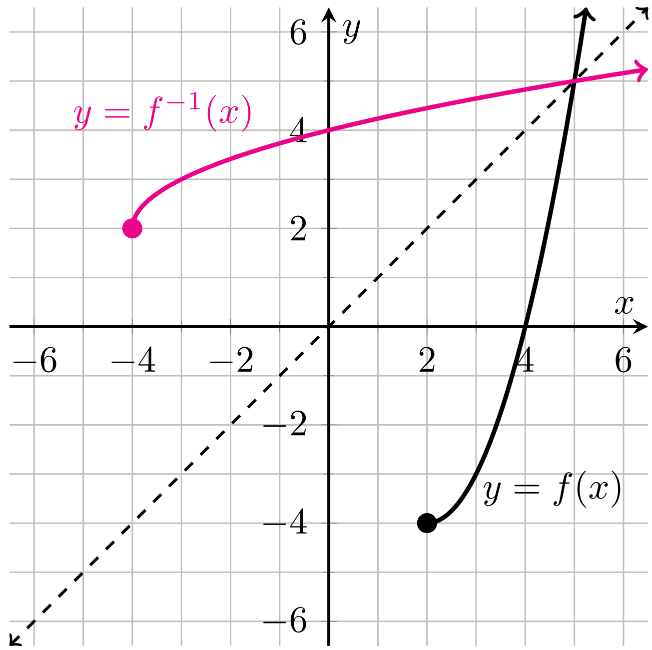
c) Find the rule for f^{-1}.

$y = (x-2)^2 - 4$

Swap x and y:

$$(y-2)^2 - 4 = x$$
$$(y-2)^2 = x + 4$$
$$y - 2 = \sqrt{x+4}$$
$$y = \sqrt{x+4} + 2$$
$$f^{-1}(x) = \sqrt{x+4} + 2$$

d) Use the graph of $y = f(x)$ shown to plot $y = f^{-1}(x)$ on the same plane.



Example 2 Find the inverse function of $g(x) = -2\sqrt{x-5}+3$, and state the domain and range for each of g and g^{-1}.

$$y = -2\sqrt{x-5} + 3$$

swap $x \leftrightarrow y$:

$$-2\sqrt{y-5} + 3 = x$$
$$-2\sqrt{y-5} = x - 3$$
$$\sqrt{y-5} = -\tfrac{1}{2}(x-3)$$
$$y - 5 = \left(-\tfrac{1}{2}(x-3)\right)^2$$
$$= \tfrac{1}{4}(x-3)^2$$
$$y = \tfrac{1}{4}(x-3)^2 + 5$$
$$g^{-1}(x) = \tfrac{1}{4}(x-3)^2 + 5$$

domain of $g = [5, \infty)$

range of $g = (-\infty, 3]$

domain of $g^{-1} = (-\infty, 3]$

range of $g^{-1} = [5, \infty)$

Example 3 Find the inverse function of $f(x) = [5(x+4)]^{1/3} - 9$.

$$y = [5(x+4)]^{1/3} - 9$$

swap $x \leftrightarrow y$:

$$[5(y+4)]^{1/3} - 9 = x$$
$$[5(y+4)]^{1/3} = x + 9$$
$$5(y+4) = (x+9)^3$$
$$y + 4 = \tfrac{1}{5}(x+9)^3$$
$$y = \tfrac{1}{5}(x+9)^3 - 4$$
$$f^{-1}(x) = \tfrac{1}{5}(x+9)^3 - 4$$

Example 4 Find the inverse function of $g(x) = -\frac{3}{4}(2x-7)^3 + 5$.

$$y = -\tfrac{3}{4}(2x-7)^3 + 5$$

swap $x \leftrightarrow y$:

$$-\tfrac{3}{4}(2y-7)^3 + 5 = x$$
$$-\tfrac{3}{4}(2y-7)^3 = x - 5$$
$$(2y-7)^3 = -\tfrac{4}{3}(x-5)$$
$$2y - 7 = \sqrt[3]{-\tfrac{4}{3}(x-5)}$$
$$2y = \sqrt[3]{-\tfrac{4}{3}(x-5)} + 7$$
$$y = \tfrac{1}{2}\sqrt[3]{-\tfrac{4}{3}(x-5)} + \tfrac{7}{2}$$
$$g^{-1}(x) = \tfrac{1}{2}\sqrt[3]{-\tfrac{4}{3}(x-5)} + \tfrac{7}{2}$$
$$= \sqrt[3]{\tfrac{1}{8}}\sqrt[3]{-\tfrac{4}{3}(x-5)} + \tfrac{7}{2}$$
$$= -\sqrt[3]{\tfrac{1}{6}(x-5)} + \tfrac{7}{2}$$

Chapter 8

Exponential and Logarithmic Functions

8.1 Exponential Functions 136
8.2 Logarithms 139
8.3 Logarithmic Functions 142
8.4 Natural Exponents and Logarithms 145
8.5 Exponential and Logarithmic Equations 149
8.6 Exponential Regression 151

8.1 Exponential Functions

An <u>exponential function</u> is a function of the form

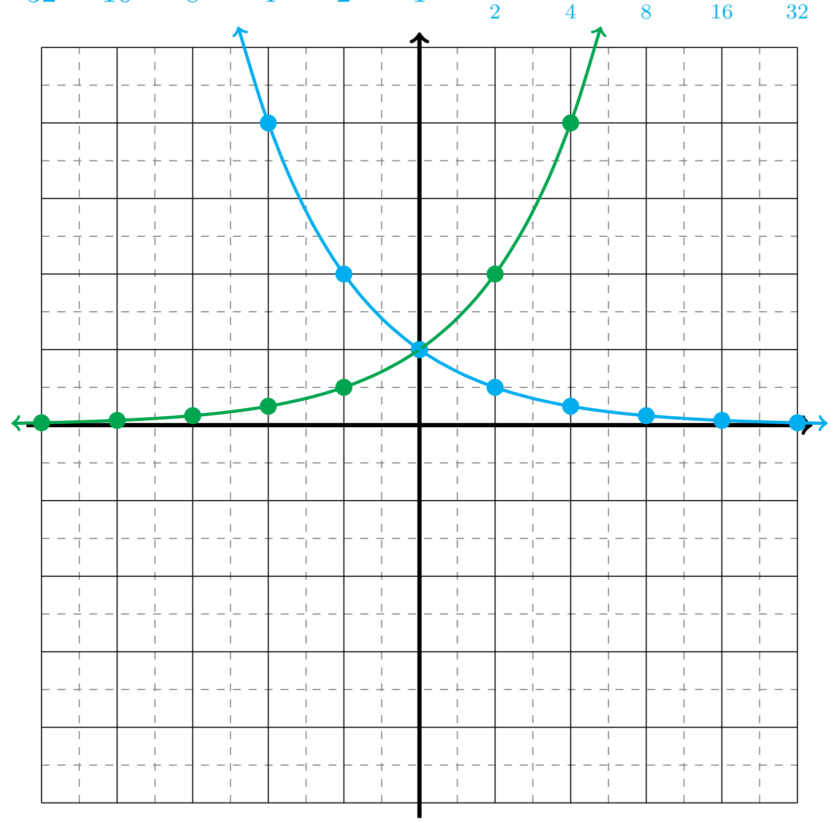
$$f(x) = A \cdot b^x + k$$

where the <u>base</u>, b, is a positive real number which is not 1. The simplest cases have $A = 1$ and $k = 0$, such as with the following two examples.

functions	$f(x) = 2^x$ $g(x) = \left(\frac{1}{2}\right)^x$
domain	$\mathbb{R}$
range	$(0, \infty)$
relation type	one-to-one
x-intercept	none
y-intercept	$(0, 1)$
horizontal asymptote	$y = 0$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(x)$	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$



For $b > 1$, including $b = 2$ above, the function shows <u>exponential growth</u>, which means as the function increases, the rate of increase is also increasing proportionally.

For $0 < b < 1$, including $b = \frac{1}{2}$ above, the function shows <u>exponential decay</u>, which means as the function decreases, the rate of decrease is also decreasing proportionally.

A sketch of an exponential function should include:

shape of curve	exponential curve showing growth or decay
x-intercept	$y = 0$, find x by solving $f(x) = 0$, may not exist
y-intercept	$x = 0$, find y by evaluating $y = f(0)$
asymptote	Horizontal: $y = k$
endpoints	evaluate the function at the bounds of the domain

It is a good idea to show an additional point, such as $(1, f(1))$, to show the rate of growth or decay.

Example 1 Sketch a graph of $f(x) = \frac{1}{2}3^x - \frac{9}{2}$.

x-intercept: $(2, 0)$

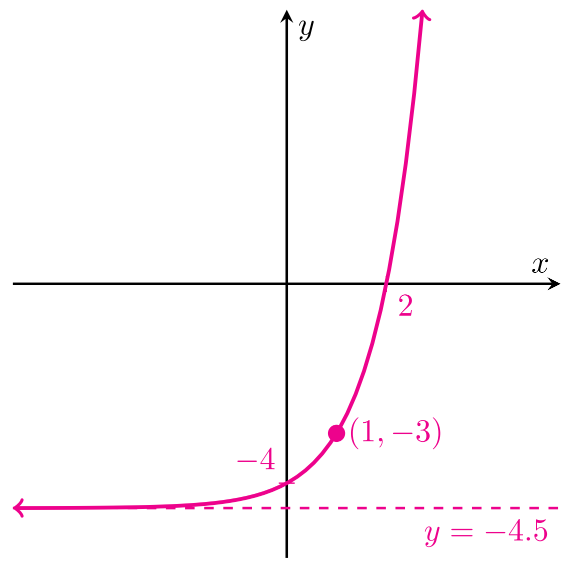
$$\begin{aligned} \tfrac{1}{2}3^x - \tfrac{9}{2} &= 0 \\ \tfrac{1}{2}3^x &= \tfrac{9}{2} \\ 3^x &= 9 \\ x &= 2 \end{aligned}$$

y-intercept: $(0, -4)$

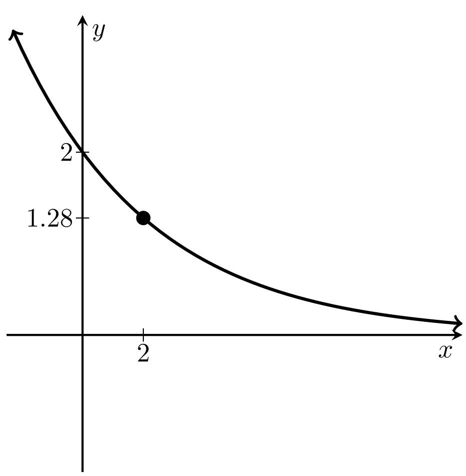
as $f(0) = \frac{1}{2} - \frac{9}{2} = -4$

asymptote: $y = -\frac{9}{2}$

endpoints: none, as domain is $\mathbb{R}$



Example 2 Identify the function g represented in the graph below.



asymptote: $y = 0 \implies k = 0$

y-intercept: $g(0) = A \cdot b^0 = 2 \implies A = 2$

point: $g(2) = 2 \cdot b^2 = 1.28$

$b^2 = 0.64 \implies b = 0.8$

$$g(x) = 2(0.8)^x$$

Example 3 Sketch a graph of $g(x) = 4^{\left(\frac{3}{2}x-1\right)} + 1$.

$$\begin{aligned} g(x) &= 4^{\left(\frac{3}{2}x-1\right)} + 1 \\ &= \left(4^{3/2}\right)^x \cdot 4^{-1} + 1 \\ &= \tfrac{1}{4} \cdot 8^x + 1 \end{aligned}$$

x-intercept: none

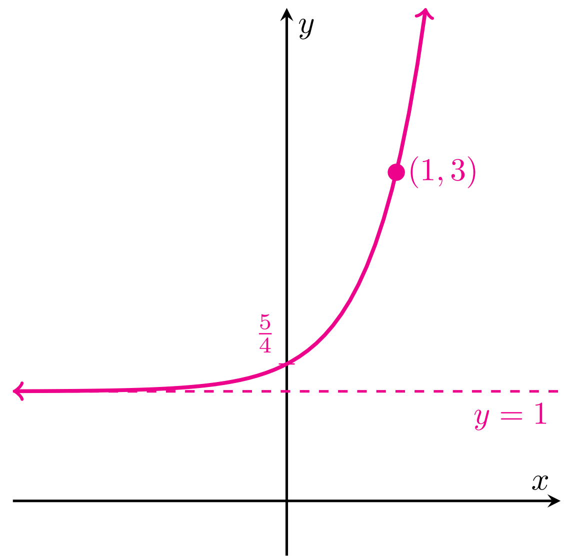
$\frac{1}{4} \cdot 8^x + 1 = 0$ has no solution

y-intercept: $(0, \frac{5}{4})$

as $g(0) = \frac{1}{4} + 1 = \frac{5}{4}$

asymptote: $y = 1$

endpoints: none, as domain is $\mathbb{R}$



Example 4 Suppose f is an exponential function, whose graph $y = f(x)$ passes through the points $(2, 2)$ and $\left(5, \frac{1}{4}\right)$, and has an asymptote $y = 0$. Find the rule for $f(x)$.

$$k = 0 \implies f(x) = Ab^x$$

$$f(2) = Ab^2 = 2$$

$$f(5) = Ab^5 = \frac{1}{4}$$

$$\frac{Ab^5}{Ab^2} = \frac{1/4}{2}$$

$$b^3 = \tfrac{1}{8}$$

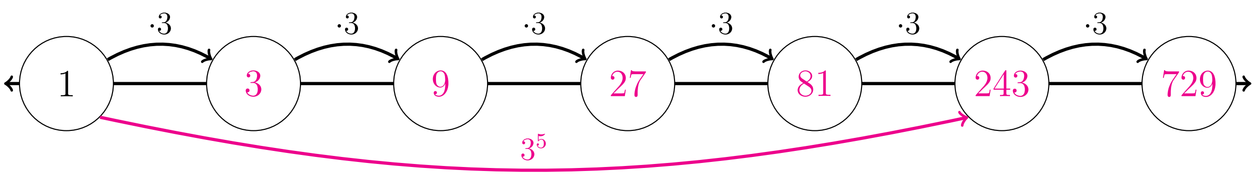
$$\begin{aligned} b &= \tfrac{1}{2} \\ A\left(\tfrac{1}{2}\right)^2 &= 2 \\ \tfrac{1}{4}A &= 2 \\ A &= 8 \\ f(x) &= 8\left(\tfrac{1}{2}\right)^x \end{aligned}$$

8.2 Logarithms

Consider the equation $3^x = 243$, whose solution is the answer to the question

Which power of 3 is 243?

The diagram illustrates that the solution is $x = 5$.



The mathematical operation which answers the question above is the logarithm. This particular case is written

$$\log_3 243 = 5$$

which is read as "the logarithm base 3 of 243." In general,

$$x = a^n \qquad \Longleftrightarrow \qquad \log_a x = n$$

Example 1

$\log_5 125 = 3$ because $5^3 = 125$

$\log_2 256 = 8$ because $2^8 = 256$

$\log_4 \frac{1}{16} = 4$ because $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$\log_7 \sqrt{7} = \frac{1}{2}$ because $7^{1/2} = \sqrt{7}$

Note that if the base is omitted, it is assumed to be 10. This is sometimes known as a common logarithm.

Example 2

$\log 10000 = 4$ because $10^4 = 10000$

$\log 0.001 = -3$ because $10^{-3} = 0.001$

Example 3 Write the following equations in logarithmic form.

$a = 3^b$	$s = t^k$	$p = 10^r$
$b = \log_3 a$	$k = \log_t s$	$r = \log p$

Example 4 Write the following equations in exponential form.

$u = \log_2 v$	$m = \log n$	$w = \log_y z$
$v = 2^u$	$n = 10^m$	$z = y^w$

Logarithm Rules

Recall that we reviewed the exponent rules in section 7.2. Some of those rules can be rewritten as equivalent logarithm rules.

Exponent Product Rule $$a^m \cdot a^n = a^{m+n}$$	**Logarithm Product Rule** $$\log_a(x \cdot y) = \log_a x + \log_a y$$
Exponent Quotient Rule $$\frac{a^m}{a^n} = a^{m-n}$$	**Logarithm Quotient Rule** $$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
Exponent Power Rule $$(a^m)^n = a^{mn}$$	**Logarithm Power Rule** $$\log_a(x^n) = n \cdot \log_a x$$
Negative Exponent Rule $$a^{-n} = \frac{1}{a^n}$$	**Reciprocal Logarithm Rule** $$\log_a\left(\frac{1}{x}\right) = -\log_a x$$
Exponent Special Values $$a^0 = 1 \qquad a^1 = a$$	**Logarithm Special Values** $$\log_a 1 = 0 \qquad \log_a a = 1$$

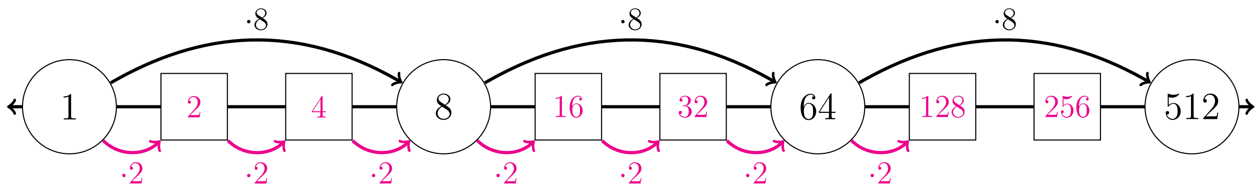
Example 5 Simplify the following without using a calculator.

$$2\log_6 3 + \log_6 4 = \log_6 3^2 + \log_6 4 = \log_6 9 + \log_6 4 = \log_6 36 = 2$$

$$\log_5 8 - \log_5 1000 = \log_5 \tfrac{8}{1000} = \log_5 \tfrac{1}{125} = -3$$

The Change of Base Rule

Recall from section 7.2 that we used the following diagram to illustrate $8^{7/3} = 128$:



We can state this in logarithmic form as $\log_8 128 = \dfrac{7}{3}$

When we originally calculated this, it was difficult to think of 128 as a power of 8. Instead, we expressed both numbers using 2 as the base, which in logarithmic form are

$$\log_2 128 = 7 \qquad \log_2 8 = 3$$

Equivalently, we can write $\log_8 128 = \dfrac{\log_2 128}{\log_2 8}$

This is an example of the following rule:

Theorem: Change of Base Rule

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Example 6 Use the change of base rule to simplify the following.

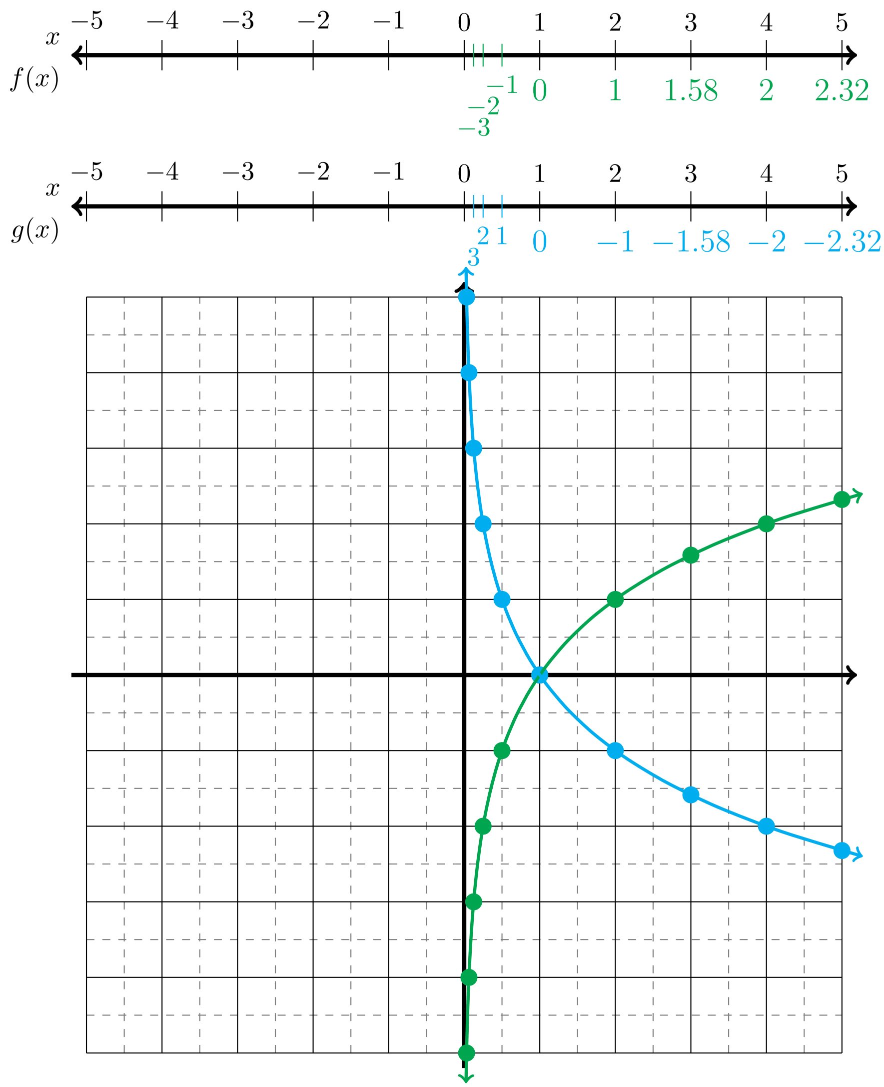
$$\log_{27} 81 = \frac{\log_3 81}{\log_3 27} = \frac{4}{3}$$

$$\log_{25} \sqrt[3]{5} = \frac{\log_5 \sqrt[3]{5}}{\log_5 25} = \frac{1/3}{2} = \frac{1}{6}$$

8.3 Logarithmic Functions

A <u>logarithmic function</u> is a function of the form $f(x) = \log_b[n(x-h)]$ where the <u>base</u>, b, is a positive real number which is not 1. The simplest cases have $n = 1$ and $h = 0$, such as with the following two examples.

functions	$f(x) = \log_2(x)$ $g(x) = \log_{0.5}(x)$
domain	$(0, \infty)$
range	$\mathbb{R}$
relation type	one-to-one
x-intercept	$(0, 1)$
y-intercept	none
vertical asymptote	$x = 0$



Example 1 Express $f(x) = \log_5(x) + 2$ in the form stated above.

$$f(x) = \log_5(x) + 2$$
$$= \log_5(x) + \log_5(25)$$
$$= \log_5(25x)$$

Example 2 Express $g(x) = \frac{1}{3}\log_2(x)$ in the form stated above.

$$g(x) = \tfrac{1}{3}\log_2(x)$$
$$= \frac{\log_2 x}{\log_2 8}$$
$$= \log_8(x)$$

A sketch of an logarithmic function should include:

shape of curve	logarithmic curve, exponential curve reflected over $y = x$
x-intercept	$y = 0$, find x by solving $f(x) = 0$
y-intercept	$x = 0$, find y by evaluating $y = f(0)$, may not exist
asymptote	vertical: $x = h$

Example 3 Sketch a graph of $f(x) = \log_2 \left[\frac{1}{3}(x-4)\right]$.

x-intercept: $(7, 0)$

$$\log_2\left[\tfrac{1}{3}(x-4)\right] = 0$$
$$\tfrac{1}{3}(x-4) = 1$$
$$x - 4 = 3 \implies x = 7$$

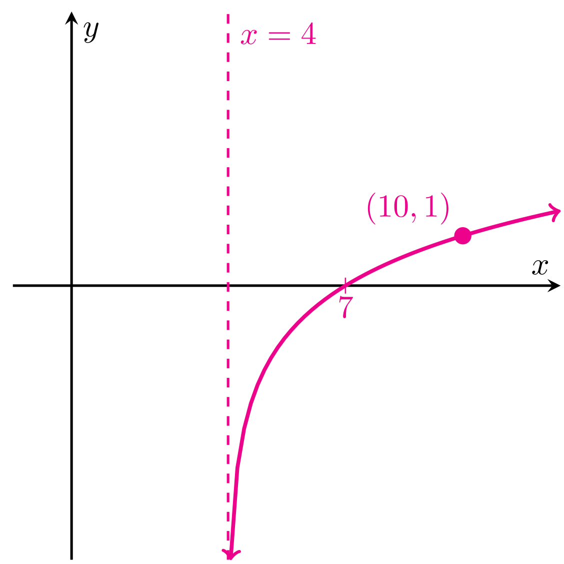
y-intercept: none

as $f(0) = \log_2(-\frac{4}{3})$ is undefined

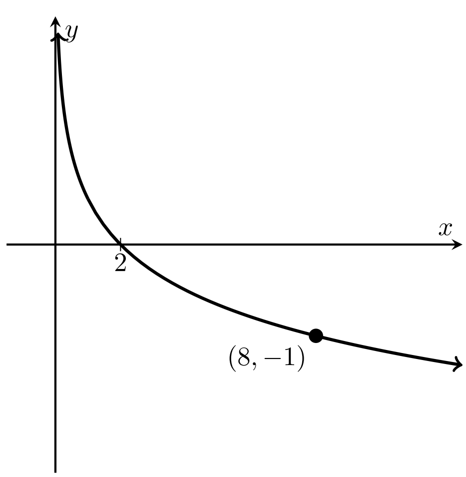
asymptote: $x = 4$

other point: $(10, 1)$

$$\log_2\left[\tfrac{1}{3}(x-4)\right] = 1$$
$$\tfrac{1}{3}(x-4) = 2$$
$$x - 4 = 6 \implies x = 10$$



Example 4 Identify the function g represented in the graph below.



asymptote: $x = 0 \implies g(x) = \log_b nx$

x-intercept: $g(2) = \log_b(n \cdot 2) = 0$

$\implies 2n = 1 \implies n = \frac{1}{2}$

point: $g(8) = \log_b 4 = -1$

$\implies b^{-1} = 4 \implies b = \frac{1}{4}$

$$g(x) = \log_{0.25}\left(\tfrac{1}{2}x\right)$$

Exponential and Logarithmic Functions as Inverses

The inverse of an exponential function is a logarithmic function with the same base.

This means that the inverse of $f(x) = a^x$ is $f^{-1}(x) = \log_a x$.

Example 5 Find the inverse function of $f(x) = 15 \cdot 3^x + 2$, and state the domain and range for each of f and f^{-1}.

$$y = 15 \cdot 3^x + 2$$

swap $x \leftrightarrow y$:

$$15 \cdot 3^y + 2 = x$$

$$15 \cdot 3^y = x - 2$$

$$3^y = \frac{x-2}{15}$$

$$y = \log_3\left(\tfrac{x-2}{15}\right)$$

$$f^{-1}(x) = \log_3\left(\tfrac{x-2}{15}\right)$$

domain of $f = \mathbb{R}$

range of $f = (2, \infty)$

domain of $f^{-1} = (2, \infty)$

range of $f^{-1} = \mathbb{R}$

Example 6 Find the inverse function of $g(x) = \log[6(x-4)]$, and state the domain and range for each of g and g^{-1}.

$$y = \log[6(x-4)]$$

swap $x \leftrightarrow y$:

$$\log[6(y-4)] = x$$

$$6(y-4) = 10^x$$

$$y - 4 = \tfrac{1}{6} \cdot 10^x$$

$$y = \tfrac{1}{6} \cdot 10^x + 4$$

$$f^{-1}(x) = \frac{1}{6} \cdot 10^x + 4$$

domain of $g = (4, \infty)$

range of $g = \mathbb{R}$

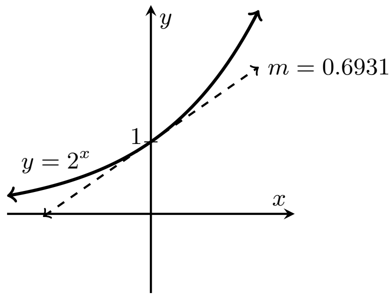
domain of $g^{-1} = \mathbb{R}$

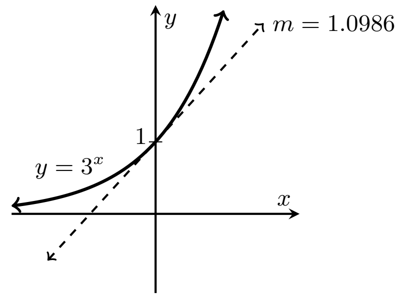
range of $g^{-1} = (4, \infty)$

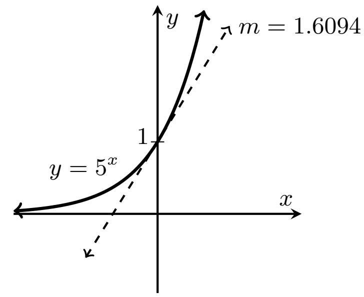
8.4 Natural Exponents and Logarithms

The Base e

Observe the following graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$.



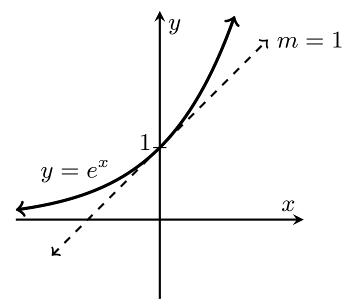




You should recall that changing the base of the exponent does not change the y-intercept, which is $(0, 1)$ for each curve. However, changing the base does change how steep the curve is at this point. This is represented by the dashed line, which is the tangent to the curve at the y-intercept.[1] Notice that the slopes of these tangents are decimal values, which each turn out to be irrational.

We might wonder if it's possible for the slope of this tangent to have an exact integer value, such as 1. As it happens, this occurs when the base is a particular irrational constant, which we denote e, and has the value

$$e = 2.71828182845904523536\ldots$$



The relationship between a function and the slopes of its tangents is the basis for much of calculus, which makes the function $f(x) = e^x$ very important. e shows up in many other areas of math also, as well as being used in science, engineering, finance and many other applications.

For Algebra 2, we need to know of the existence of e and that it is closely related to exponents and logarithms. However, we don't need to worry if we don't yet understand why it is important or where it comes from.

When exponents or logarithms have e as their base, they are called natural. All exponential and logarithmic functions can be written as transformations of natural exponents and logarithms, so we can use these as parent functions.

The natural logarithm is important enough that it gets its own notation:

$$\ln(x) = \log_e(x)$$

[1]Remember from Geometry that the tangent to a circle is a straight line which touches the circle at a single point? Graphs of functions also have tangents, which have a very similar meaning.

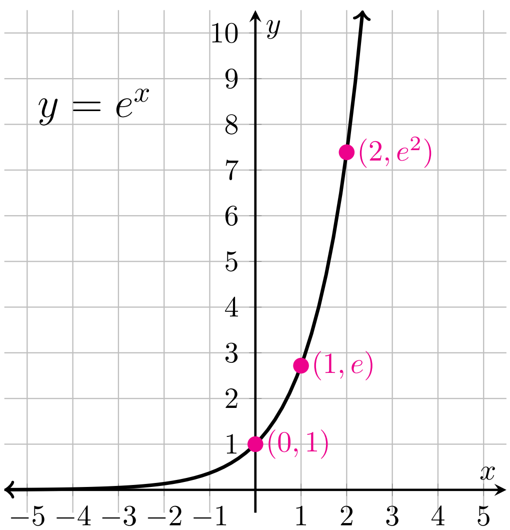
Natural Exponents

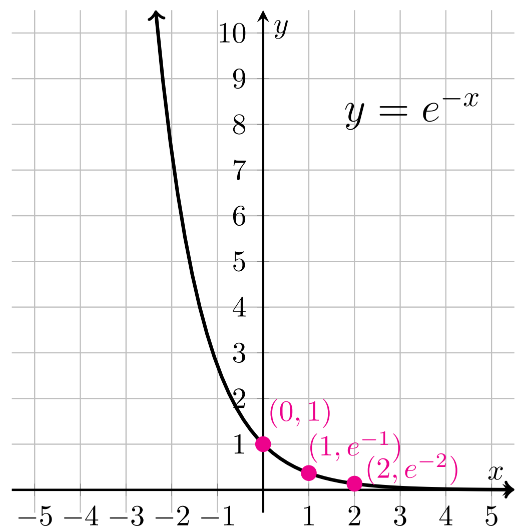
The parent function for natural exponents is $f(x) = e^x$, which leads to the general form

$$f(x) = Ae^{nx} + k$$

Instead of changing the base to control the rate of exponential growth or decay, we can change the value of n. If n is positive, the function exhibits exponential growth. If n is negative, the function exhibits exponential decay.

Example 1 Plot the points at $x = 0, 1, 2$ on each of the following graphs, and label them with exact coordinates.





Since e^x and $\ln x$ are inverses, we can use the result $e^{\ln a} = a$ to change the base of an exponent to e:

$$a^x = \left(e^{\ln a}\right)^x = e^{\ln(a) \cdot x}$$

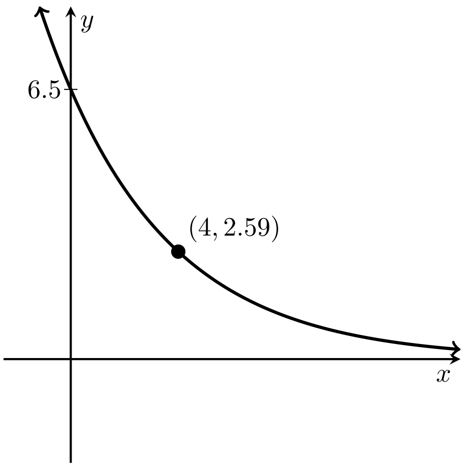
Example 2 Express $f(x) = 5 \cdot 4^x$ using e as the base.

$$\begin{aligned} f(x) &= 5 \cdot 4^x \\ &= 5 \cdot \left(e^{\ln(4)}\right)^x \\ &= 5e^{\ln(4)x} \end{aligned}$$

Example 3 Express $g(x) = 3 \cdot \left(\frac{1}{8}\right)^x$ as a natural exponential function.

$$\begin{aligned} g(x) &= 3 \cdot \left(\tfrac{1}{8}\right)^x \\ &= 3 \cdot 8^{-x} \\ &= 3 \cdot \left(e^{\ln(8)}\right)^{-x} \\ &= 3e^{-\ln(8)x} \end{aligned}$$

Example 4 Identify the function f represented in the graph below.



asymptote is $y = 0 \implies f(x) = Ae^{nx}$

$f(0) = A = 6.5$

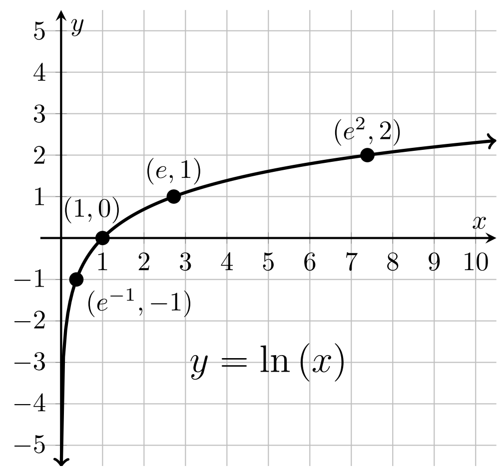
$f(4) = 6.5e^{4n} = 2.59$

$e^{4n} = \frac{2.59}{6.5} \implies 4n = \ln\left(\frac{2.59}{6.5}\right)$

$n = \frac{1}{4}\ln\left(\frac{2.59}{6.5}\right) = -0.23$

$f(x) = 6.5e^{-0.23x}$

Natural Logarithms



The parent function for natural logarithms is $f(x) = \ln x$, which leads to the general form

$$f(x) = A \cdot \ln\left[n\left(x - h\right)\right]$$

Instead of changing the base to control the direction and shape of the logarithmic curve, we can change the value of A.

We already have the change of base rule which we can use to change logarithms to their natural form:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

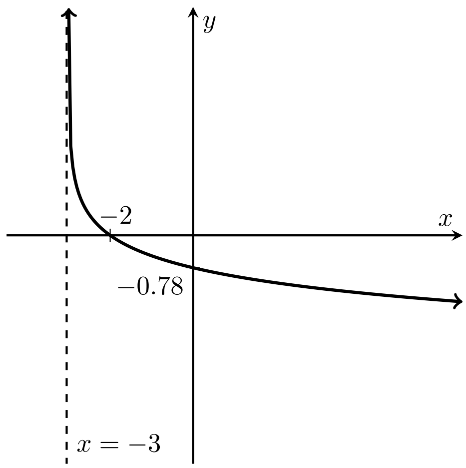
Example 5 Express $f(x) = \log_4 3x$ using the natural logarithm.

$$f(x) = \log_4(3x) = \frac{1}{\ln 4}\ln(3x)$$

Example 6 Express $g(x) = \log_{0.2} x$ using the natural logarithm.

$$g(x) = \log_{0.2} x = \frac{1}{\ln 0.2}\ln x = -\frac{1}{\ln 5}\ln x$$

Example 7 Identify the function g represented in the graph below.



asymptote is $x = -3$

$$\implies g(x) = A\ln\left[n\left(x+3\right)\right]$$

$$g(-2) = A\ln\left[n\left(-2+3\right)\right] = 0$$

$$\implies \ln\left(n\right) = 0$$

$$\implies n = e^0 = 1$$

$$\implies g(x) = A\ln\left(x+3\right)$$

$$g(0) = A\ln 3 = -0.78$$

$$\implies A = \frac{-0.78}{\ln 3} = -0.71$$

$$g(x) = -0.71\ln\left(x+3\right)$$

Example 8 Find the inverse function of $f(x) = 20e^{-0.001x} + 5$. State the domain and range of each f and f^{-1}.

$$\begin{aligned} y &= 20e^{-0.01x} + 5 \\ \text{swap } x, y: \quad x &= 20e^{-0.01y} + 5 \\ 20e^{-0.01y} &= x - 5 \\ e^{-0.01y} &= 0.05\left(x-5\right) \\ -0.01y &= \ln\left[0.05\left(x-5\right)\right] \\ y &= -100\ln\left[0.05\left(x-5\right)\right] \\ f^{-1}(x) &= -100\ln\left[0.05\left(x-5\right)\right] \end{aligned}$$

domain of $f = \mathbb{R}$, range of $f = (5, \infty)$

domain of $f^{-1} = (5, \infty)$, range of $f^{-1} = \mathbb{R}$

8.5 Exponential and Logarithmic Equations

Method 1: Equating the Base

The simplest method to solve equations involving exponents or logarithms is often to write both sides with the same base. Then we can use the following theorem.

Theorem

Two exponential expressions with the same base are equal iff (if and only if) they have the same exponent.

Example 1 Solve $81^{2x+1} = \sqrt{3}$.

$$\begin{aligned}(3^4)^{2x+1} &= 3^{1/2}\\ 3^{4(2x+1)} &= 3^{1/2}\\ 4(2x+1) &= \tfrac{1}{2}\\ 2x+1 &= \tfrac{1}{8}\\ 2x &= -\tfrac{7}{8}\\ x &= -\tfrac{7}{16}\end{aligned}$$

Example 2 Solve $6^{5x+3} = 36^{4x+9}$.

$$\begin{aligned}6^{5x+3} &= (6^2)^{4x+9}\\ &= 6^{8x+18}\\ 5x+3 &= 8x+18\\ -3x &= 15\\ x &= -5\end{aligned}$$

This applies equally to logarithms, as they are the inverse of exponents. You'll need to check for extraneous solutions.

Example 3 Solve $\log(4x-2) - \log(x-5) = 1$.

$$\begin{aligned}\log\left(\frac{4x-2}{x-5}\right) &= \log 10\\ \frac{4x-2}{x-5} &= 10\\ 4x-2 &= 10x-50\\ -6x &= -48\\ x &= 8\end{aligned}$$

Example 4 Solve $2\ln(x) = \ln(2x+3)$.

$$\begin{aligned}\ln(x^2) &= \ln(2x+3)\\ x^2 &= 2x+3\\ x^2-2x-3 &= 0\\ (x-3)(x+1) &= 0\\ x = 3 \text{ or } x &= -1\end{aligned}$$

$\ln(-1)$ is undefined

$\implies x = 3$

Method 2: Using Inverse Operations

Since exponents and logarithms are inverses of each other, we can use them to solve equations involving the other. The solutions obtained when using this method are often irrational.

Example 5 Solve $\log_3(x+9) = 2$.

$$\begin{aligned} \log_3(x+9) &= 2 \\ x+9 &= 3^2 \\ &= 8 \\ x &= -1 \end{aligned}$$

Example 6 Solve $3e^{x/4} + 4 = 10$ exactly.

$$\begin{aligned} 3e^{x/4} + 4 &= 10 \\ 3e^{x/4} &= 6 \\ e^{x/4} &= 2 \\ \tfrac{x}{4} &= \ln 2 \\ x &= 4\ln 2 \end{aligned}$$

Example 7 Solve $4^{2x-3} = 20$ to 2 decimal places.

$$\begin{aligned} 4^{2x-3} &= 20 \\ 2x-3 &= \log_4 20 \\ 2x &= \log_4 20 + 3 \\ x &= \tfrac{1}{2}(\log_4 20 + 3) \\ &\approx 2.58 \end{aligned}$$

Example 8 Solve $2\ln(x-1) + 5 = 1$ to 3 decimal places.

$$\begin{aligned} 2\ln(x-1) + 5 &= 1 \\ 2\ln(x-1) &= -4 \\ \ln(x-1) &= -2 \\ x-1 &= e^{-2} \\ x &= e^{-2} + 1 \\ &\approx 1.135 \end{aligned}$$

Method 3: Using a Substitution

Sometimes we can change an equation to a simplified form using a thoughtful substitution.

Example 9 Solve $3^{2x} - 6 \cdot 3^x - 27 = 0$.

$$(3^x)^2 - 6 \cdot 3^x - 27 = 0$$

Let $a = 3^x$

$$\begin{aligned} a^2 - 6a - 27 &= 0 \\ (a-9)(a+3) &= 0 \end{aligned}$$

$a = 9$ or $a = -3$

$3^x = 9$ or $\cancel{3^x = -3}$

$x = 2$

8.6 Exponential Regression

Recall that regression is the process of fitting a modeling function to a set of data in order to approximate the relationship between variables.

Exponential regression uses an exponential function for the modelling function. This means choosing values for a and b so that $f(x) = a \cdot b^x$ fits the data as well as possible.

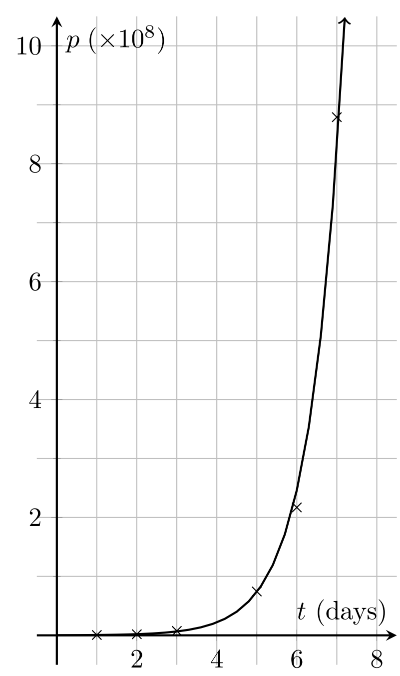
Like linear and quadratic regression, performing exponential regression involves calculating the the coefficient of determination, denoted by R^2, which measures how well the regression curve fits the data.

If your device or software offers "log mode" for this type of regression, this generally provides a better fit. Some devices do this by default.[2]

Example 1 A research lab is investigating the population of a sample of bacteria. After leaving the sample for 24 hours at a time, the number of bacteria is estimated and recorded. Let t be the number of days after the beginning of the experiment.

t (days)	1	2	3	5	6	7
p	5.74×10^5	1.85×10^6	7.49×10^6	7.43×10^7	2.17×10^8	8.79×10^8

Use exponential regression to model bacteria population.



$a = 175140, b = 3.34699, R^2 = 0.999$

$$p(t) = 175140(3.34699)^t$$

The model is a very good fit for the data, as R^2 is close to 1.

Example 2 Predict the population at the beginning of the experiment.

$p(0) = 175140(3.34699)^0 = 175140$

Example 3 The researchers weren't able to collect data on day 4. Estimate what the population would have been that day.

$p(4) = 175140(3.34699)^4 = 2.20 \times 10^7$

[2]How this works, and the reasons why performing exponential regression this way is preferable, are beyond the scope of Algebra 2.

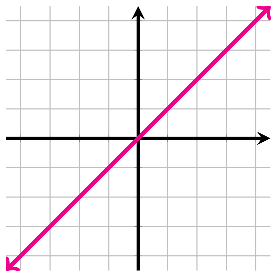
Chapter 9

Further Functions

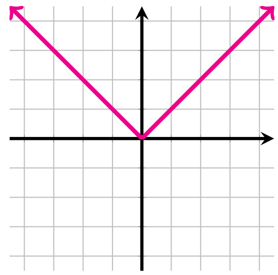
9.1 Identifying Functions 154
9.2 Algebraic Combinations of Functions 156
9.3 Function Composition 158
9.4 Piecewise Functions 161

9.1 Identifying Functions

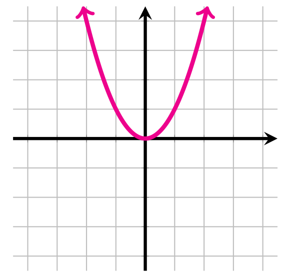
Review of Parent Functions



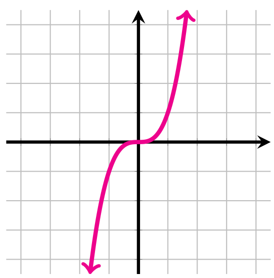
$f(x) = x$



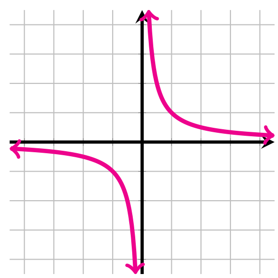
$f(x) = |x|$



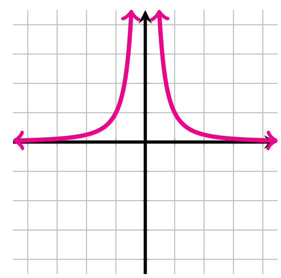
$f(x) = x^2$



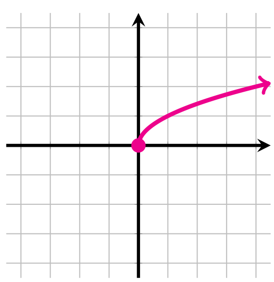
$f(x) = x^3$



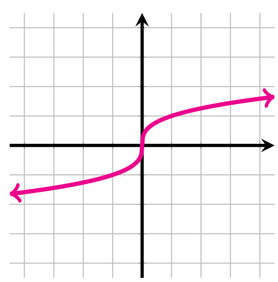
$f(x) = \frac{1}{x}$



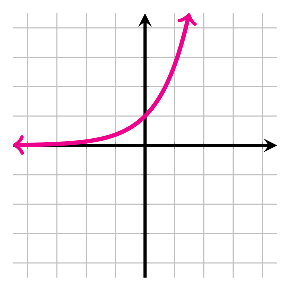
$f(x) = \frac{1}{x^2}$



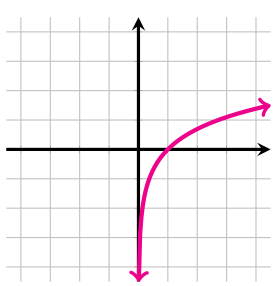
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



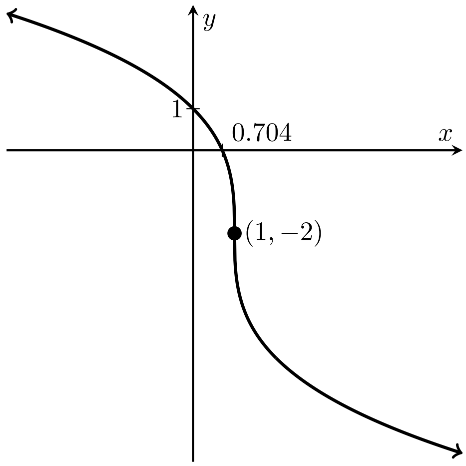
$f(x) = e^x$



$f(x) = \ln x$

Recall that we can use these parent functions, together with transformations, to construct functions. By identifying these in a graph, we can identify the corresponding function.

Example 1 Identify the function f represented in the graph below.



shape $\Longrightarrow$ cube root function

$$f(x) = A\sqrt[3]{x-h}+k$$

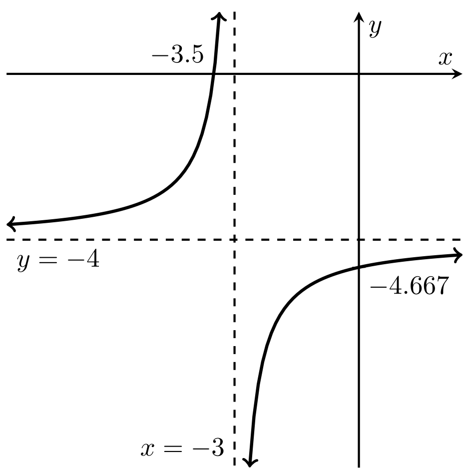
pt. of inflection $\Longrightarrow$ $f(x) = A\sqrt[3]{x-1}-2$

$f(0) = A\sqrt[3]{-1}-2 = -A-2 = 1$

$-A = 3 \Longrightarrow A = -3$

$$f(x) = -3\sqrt[3]{x-1}-2$$

Example 2 Identify the function g represented in the graph below.



hyperbola $\Longrightarrow$ $g(x) = \frac{A}{x-h}+k$

asymptotes $\Longrightarrow$ $g(x) = \frac{A}{x+3}-4$

$g(-3.5) = \frac{A}{-0.5}-4 = 0$

$\frac{A}{-0.5} = 4 \Longrightarrow A = -2$

$g(x) = \frac{-2}{x+3}-4$

9.2 Algebraic Combinations of Functions

By combining functions in a variety of ways, we can create new functions. The simplest thing we can do is to add, subtract or multiply functions.

- If $h = f + g$, then $h(x) = f(x) + g(x)$ for each value of x.
- If $h = f - g$, then $h(x) = f(x) - g(x)$ for each value of x.
- If $h = f \cdot g$, then $h(x) = f(x)g(x)$ for each value of x.

Note that for each of these cases, $h(x)$ is only defined if both $f(x)$ and $g(x)$ are defined. This means that the domain of h is the intersection of the domains of f and g.

We can also divide functions.

- If $h = f/g$, then $h(x) = \dfrac{f(x)}{g(x)}$ for each value of x.

In this case, we need to remember that we can't divide by zero. So $h(x)$ is only defined if both $f(x)$ and $g(x)$ are defined, and $g(x) \neq 0$.

Example 1 Complete the table.

x	-2	-1	0	1	2	3	4
$f(x)$	undef	2	6	0	1	3	-2
$g(x)$	3	0	2	4	undef	1	-2
$(f+g)(x)$	undef	2	8	4	undef	4	-4
$(f-g)(x)$	undef	2	4	-4	undef	2	0
$(f \cdot g)(x)$	undef	0	12	0	undef	3	4
$(f/g)(x)$	undef	undef	3	0	undef	3	1

Example 2 State the domains of all of the functions in example 1.

domain of $f = \{-1, 0, 1, 2, 3, 4\}$

domain of $g = \{-2, -1, 0, 1, 3, 4\}$

domain of $(f+g)$ = domain of $(f-g)$ = domain of $(f \cdot g) = \{-1, 0, 1, 3, 4\}$

domain of $(f/g) = \{0, 1, 3, 4\}$

Example 3 State the rule for $h = f + g$ if $f(x) = \ln(x+3)$ and $g(x) = \dfrac{1}{x-5}$. Find the domains of f, g and h.

$$h(x) = \ln(x+3) + \frac{1}{x-5}$$

domain of $f = (-3, \infty)$

domain of $g = \mathbb{R} \setminus \{5\}$

domain of $h = (-3, 5) \cup (5, \infty)$

In the previous example, the domain of the combined function could be identified from its rule as the implied domain.

In the following examples, we'll find that the domain of the combined function is different from the domain implied by its rule.

Example 4 Find and simplify the rule for $w = u \cdot v$ if $u(x) = \dfrac{1}{x+1}$ and $v(x) = x^3 + 3x^2 + 3x + 1$. Find the domains of u, v and w.

$$\begin{aligned} w(x) &= u(x)v(x) \\ &= \frac{1}{x+1}(x^3 + 3x^2 + 3x + 1) \\ &= \frac{1}{x+1}(x+1)^3 \\ &= (x+1)^2 \end{aligned}$$

domain of $u = \mathbb{R} \setminus \{-1\}$

domain of $v = \mathbb{R}$

domain of $w = \mathbb{R} \setminus \{-1\}$

(implied domain is $\mathbb{R}$)

Example 5 Find and simplify the rule for $h = f/g$ if $f(x) = (x+3)e^{-x}$ and $g(x) = x^2 - 4x - 21$. Find the domains of f, g and h.

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ &= \frac{(x+3)e^{-x}}{x^2 - 4x - 21} \\ &= \frac{(x+3)e^{-x}}{(x+3)(x-7)} \\ &= \frac{e^{-x}}{x-7} \end{aligned}$$

domain of $f = \mathbb{R}$

domain of $g = \mathbb{R}$

domain of $h = \mathbb{R} \setminus \{-3, 7\}$

(implied domain is $\mathbb{R} \setminus \{7\}$)

9.3 Function Composition

Another way to combine functions is composition, which means using the output of one function as the input of another. The composition of f and g is denoted $f \circ g$, and the function is defined as

$$(f \circ g)(x) = f\left[g\left(x\right)\right]$$

Note that the order matters, because switching f and g results in a different function.

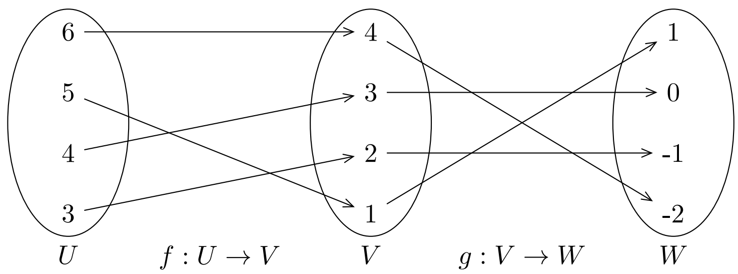
$$(g \circ f)(x) = g\left[f\left(x\right)\right]$$

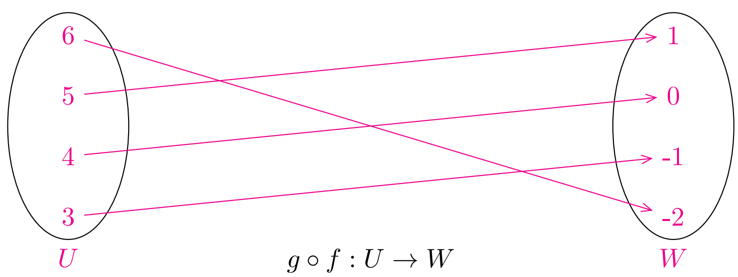
Example 1

a) Complete the mapping diagram for $g \circ f$.

b) Are there any values for which $f \circ g$ is defined?

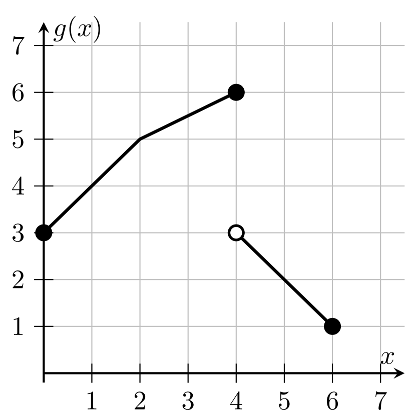
No, the range of g and the domain of f share no values, so $f\left[g\left(x\right)\right]$ is always undefined.





Example 2 Use the function definitions to evaluate the compositions.

x	$f(x)$
0	4
1	3
2	0
3	1
4	5
5	6
6	2



$$\begin{aligned}(f \circ g)(5) &= f\left[g\left(5\right)\right] \\ &= f(2) \\ &= 0\end{aligned}$$

$$\begin{aligned}(g \circ f)(3) &= g\left[f\left(3\right)\right] \\ &= g(1) \\ &= 4\end{aligned}$$

$$(f \circ g)(6) = f[g(6)] = f(1) = 3$$

$$(g \circ f)(2) = g[f(2)] = g(0) = 3$$

$$(g \circ g)(2) = g[g(2)] = g(5) = 2$$

$$(f \circ f)(0) = f[f(0)] = f(4) = 5$$

$$(g \circ g)(3) = g[g(5.5)] = g(5.5) = 1.5$$

$(f \circ g)(3) = f[g(3)] = f(5.5)$ is undefined

Example 3 $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$. Find $g \circ f$ and $f \circ g$.

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= g(x^2 + 2x) \\ &= 3(x^2 + 2x) - 5 \\ &= 3x^2 + 6x - 5\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f(3x - 5x) \\ &= (3x - 5)^2 + 2(3x - 5) \\ &= 9x^2 - 30x + 25 \\ &\quad + 6x - 10 \\ &= 9x^2 - 24x + 15\end{aligned}$$

Example 4 $f : [-3, 6] \to \mathbb{R}$ where $f(x) = x^2$, and $g : (0, 11) \to \mathbb{R}$ where $g(x) = x - 7$. Find $f \circ g$, and find its domain and range.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f(x - 7) \\ &= (x - 7)^2\end{aligned}$$

If x is in the domain of $f \circ g$, then $g(x)$ is in the domain of f:

$$\begin{aligned}-3 \le g(x) \le 6 \\ -3 \le x - 7 \le 6 \\ 4 \le x \le 13\end{aligned}$$

Also, x must be in the domain of g:

$$\text{domain of } f \circ g = [4, 13] \cap (0, 11) = [4, 11)$$

$f \circ g$ has a vertex at $(7, 0)$, and endpoints at $(4, 9)$ and $(11, 16)$

$$\text{range of } f \circ g = [0, 16)$$

Composition with the Inverse

With composition, we can show that two functions are inverses, using the following theorem.

Theorem

$$f : A \to B \text{ and } f^{-1} : B \to A \text{ are } \underline{\text{inverse}} \text{ functions}$$
$$\text{iff } (f^{-1} \circ f)(x) = f^{-1}\left[f(x)\right] = x \text{ for every } x \in A$$
$$\text{and } (f \circ f^{-1})(x) = f\left[f^{-1}(x)\right] = x \text{ for every } x \in B$$

Example 5 Show that $f(x) = 5e^x - 8$ and $g(x) = \ln\left[\frac{1}{5}(x+8)\right]$ are inverses.

$$\begin{aligned} g\left[f(x)\right] &= g\left(5e^x - 8\right) \\ &= \ln\left[\tfrac{1}{5}\left(5e^x - 8 + 8\right)\right] \\ &= \ln\left[\tfrac{1}{5}\left(5e^x\right)\right] \\ &= \ln\left(e^x\right) \\ &= x \end{aligned}$$

Example 6 Show that $f : [4, \infty) \to \mathbb{R}$ where $f(x) = x^2 - 8x + 21$ and $g(x) = \sqrt{x-5} + 4$ are inverses.

$$\begin{aligned} f\left[g(x)\right] &= f\left(\sqrt{x-5} + 4\right) \\ &= \left(\sqrt{x-5} + 4\right)^2 - 8\left(\sqrt{x-5} + 4\right) + 21 \\ &= \sqrt{x-5}^2 + 2 \cdot 4\sqrt{x-5} + 4^2 - 8\sqrt{x-5} - 32 + 21 \\ &= x - 5 + 8\sqrt{x-5} + 16 - 8\sqrt{x-5} - 32 + 21 \\ &= x \end{aligned}$$

9.4 Piecewise Functions

We previously discussed piecewise functions in section 2.5, but only considered functions with linear pieces. In general, any function can be a piece of a piecewise function. For this course, we'll include quadratic and exponential pieces.

Example 1 Evaluate each of the following using the function f.

$$f(x) = \begin{cases} x^2+2 & 0 \le x < 3 \\ 16 \cdot 2^{-x} & 3 \le x < 6 \\ -x+11 & 6 \le x < 10 \end{cases}$$

$f(1) = (1)^2 + 2$
$= 3$

$f(8) = -8 + 11$
$= 3$

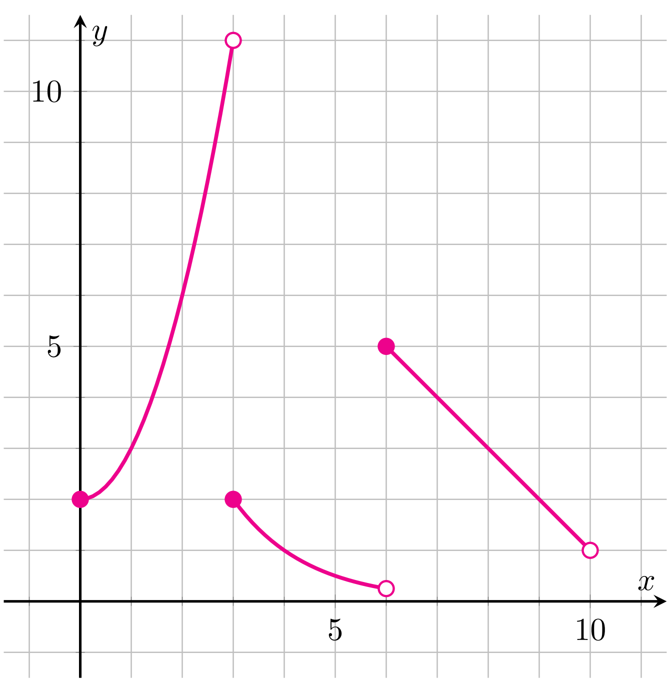
$f(5) = 16 \cdot 2^{-5}$
$= 0.5$

$f(6) = -(6) + 11$
$= 5$

$f(3) = 16 \cdot 2^{-3}$
$= 2$

$f(10)$ is undefined

Example 2 For function f above, plot its graph and find its domain and range.



Domain:

$[0,3) \cup [3,6) \cup [6,10)$
$= [0,10)$

Range:

$[2,11) \cup \left(\frac{1}{4}, 2\right] \cup (1,5]$
$= \left(\frac{1}{4}, 11\right)$

Example 3 Consider the function g defined as

$$g(x) = \begin{cases} x^2 - 8x + 12 & 1 < x \leq 5 \\ -3 & 5 < x < 8 \\ -x^2 + 20x - 99 & 8 \leq x \leq 13 \end{cases}$$

a) Find the zeros of g.

For $x \in (1, 5]$,

$$x^2 - 8x + 12 = 0$$
$$(x-2)(x-6) = 0$$
$$x = 2 \text{ or } \cancel{x = 6}$$

For $x \in (5, 8)$, $-3 \neq 0$

For $x \in [8, 13)$,

$$-x^2 + 20x - 99 = 0$$
$$-(x-9)(x-11) = 0$$
$$x = 9 \text{ or } x = 11$$

Zeros are $2, 9, 11$

b) Find the intervals g is increasing, decreasing, or constant.

For $x \in (1, 5]$, parabola is upright with a vertex at $x = 4$.

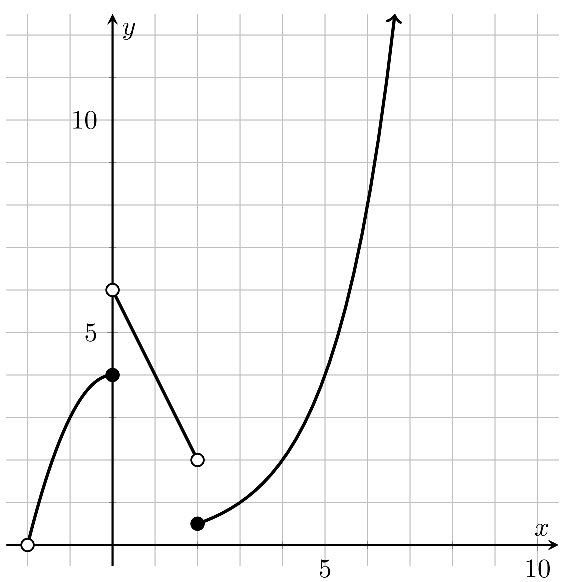
For $x \in [8, 13)$, parabola is inverted with a vertex at $x = 10$.

Increasing on $(4, 5) \cup (8, 10)$

Constant on $(5, 8)$

Decreasing on $(1, 4) \cup (10, 13)$

Example 4 Find the function h represented in the graph below.



Quadratic: Vertex at $(0, 4)$, passes through $(-2, 0) \implies y = -x^2 + 4$.

Linear: $m = -2, b = 6 \implies y = -2x + 6$

Exponential: doubling $\implies b = 2$, passes through $(3, 1) \implies y = \frac{1}{8} \cdot 2^x$

$$g(x) = \begin{cases} -x^2 + 4 & -2 < x \leq 0 \\ -2x + 6 & 0 < x < 2 \\ \frac{1}{8} \cdot 2^x & 8 \leq x \leq 13 \end{cases}$$

Chapter 10

Matrices

10.1 Matrix Operations 164
10.2 Solving Linear Systems with Matrices 166

10.1 Matrix Operations

A matrix (plural matrices) consists of numbers arranged into rows and columns in a rectangle. It is typical to assign them upper case variables, and to surround them with brackets.[1]

For example,

$$A = \begin{bmatrix} 3 & 7 & -2 \\ 9 & -4 & 1 \end{bmatrix}$$

The dimensions of a matrix denote the number of rows, m, by the number of columns, n, which we write as $m \times n$, and read as "m by n".

For example, the dimensions of A above are 2×3, or we say A is a 2×3 matrix.

The individual elements of a matrix are denoted by $a_{i,j}$, where a is the lower case letter corresponding to the matrix variable, i indicates which row, and j indicates which column.

Example 1 Write the following using A above.

$a_{1,2} = 7$ $a_{2,1} = 9$ $a_{1,3} = -2$

A matrix with the same number of rows and columns, or an $n \times n$ matrix, is called a square matrix.

An identity matrix is a square matrix with ones along its diagonal (top-left to bottom-right), and zeros everywhere else. If the identity matrix is $n \times n$, it is denoted I_n.

Example 2 Write down I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3 If $B = I_7$, find $b_{4,2}$ and $b_{5,5}$.

Because the row and column don't match, $b_{4,2}$ is not on the diagonal, so $b_{4,2} = 0$.

Meanwhile, $b_{5,5}$ is on the diagonal, so $b_{5,5} = 1$.

[1] Some mathematicians prefer to use parentheses.

Adding and Subtracting Matrices

Matrices can be added or subtracted by adding or subtracting individual elements in corresponding positions. This is only possible if the matrices have the same dimensions, and the resulting matrix will also have the same dimensions.

Example 4 If $C = \begin{bmatrix} 3 & 6 \\ -5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -7 & 8 \\ 2 & -4 \end{bmatrix}$, find $C + D$ and $C - D$.

$$C + D = \begin{bmatrix} -4 & 14 \\ -3 & -3 \end{bmatrix} \qquad C - D = \begin{bmatrix} 10 & -2 \\ -7 & 5 \end{bmatrix}$$

Multiplying a Matrix and a Scalar

To distinguish them from matrices, individual numbers are called scalars.

A scalar cannot be added to or subtracted from a matrix, but it can be multiplied. To do so, we multiply each element in the matrix by the scalar. The result is a matrix with the same dimensions as the original matrix.

Example 5 Using $A = \begin{bmatrix} 3 & 7 & -2 \\ 9 & -4 & 1 \end{bmatrix}$, find $-5A$.

$$-5A = \begin{bmatrix} -15 & -21 & 10 \\ -45 & 20 & -5 \end{bmatrix}$$

Example 6 Find $3D - 4C$, using C and D above.

$$\begin{aligned} 3D - 4C &= \begin{bmatrix} -21 & 24 \\ 6 & -12 \end{bmatrix} + \begin{bmatrix} -12 & -24 \\ 20 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -33 & 0 \\ 26 & -16 \end{bmatrix} \end{aligned}$$

10.2 Solving Linear Systems with Matrices

We can take a system of linear equations at write them as a single matrix equation:

$$\begin{cases} a_{1,1}x & + & a_{1,2}y & + & a_{1,3}z & = & b_1 \\ a_{2,1}x & + & a_{2,2}y & + & a_{2,3}z & = & b_2 \\ a_{3,1}x & + & a_{3,2}y & + & a_{3,3}z & = & b_3 \end{cases} \qquad \longleftrightarrow \qquad AX = B$$

$$\text{where} \qquad A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then we can solve the matrix equation. The techniques used are beyond the scope of this course, and tedious to perform by hand anyway, but are simple for a calculator.

Reduced Row Echelon Form

Step 1: Write matrices A and B together, which is called an augmented matrix.

$$\left[A \mid B\right] = \left[\begin{array}{ccc|c} a_{1,1} & a_{1,2} & a_{1,3} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & b_3 \end{array}\right]$$

Step 2: Apply the operation rref to the matrix using a calculator. This applies a series of operations which are equivalent to solving the system using the elimination method.

Step 3: Interpret the solution from the resulting matrix.

Example 1 Solve

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 11 \\ -x + 3y + 4z = 8 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 11 \\ -1 & 3 & 4 & 8 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$x = 3 \quad y = 1 \quad z = 2$$

Notice that A has been replaced with the identity matrix. This will always happen if there is a unique solution to the system. If not, then the matrix takes a different form.

Example 2 Solve

$$\begin{cases} 5x - 3y + z = -5 \\ 2x + y + 3z = 9 \\ 7x - 2y + 4z = 12 \end{cases}$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 1 & -5 \\ 2 & 1 & 3 & 9 \\ 7 & -2 & 4 & 12 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 10/11 & 0 \\ 0 & 1 & 13/11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Last line implies $0 = 1$, which is impossible, so no solution.

Example 3 Solve

$$\begin{cases} 5x - 3y + z = -5 \\ 2x + y + 3z = 9 \\ 7x - 2y + 4z = 4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 1 & -5 \\ 2 & 1 & 3 & 9 \\ 7 & -2 & 4 & 4 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 10/11 & 2 \\ 0 & 1 & 13/11 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is still consistent, but doesn't specify a unique solution, so infinitely many solutions.

Determinants

An important property of a square matrix is its determinant. It is denoted by vertical lines replacing the brackets around the matrix. The determinant of a matrix A can be written $|A|$ or $\det(A)$.

Determinant

The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant can be found for larger $n \times n$ matrices, but becomes much more complicated. It is much easier to find using a calculator.

Example 4 Find the following determinants.

$$\begin{vmatrix} -3 & 2 \\ 4 & -1 \end{vmatrix} = -3(-1) - 2\cdot 4$$
$$= 3 - 8$$
$$= -5$$

$$\begin{vmatrix} -1 & -4 \\ 3 & 2 \end{vmatrix} = -1\cdot 2 - (-4)\cdot 3$$
$$= -2 + 12$$
$$= 10$$

The following result is particularly useful for linear systems.

Theorem

A linear system, written in the matrix form $AX = B$,

has a unique solution iff

$$|A| \neq 0$$

Example 5 Confirm the nature of the solutions for the systems in the earlier examples.

For example 1:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ -1 & 3 & 4 \end{vmatrix} = -19 \qquad \Longrightarrow \qquad \text{unique solution}$$

For examples 2 and 3:

$$\begin{vmatrix} 5 & -3 & 1 \\ 2 & 1 & 3 \\ 7 & -2 & 4 \end{vmatrix} = 0 \qquad \Longrightarrow \qquad \text{no unique solution}$$

Chapter 11

Sequences and Series

11.1 Introduction to Sequences and Series 170
11.2 Arithmetic Sequences and Series 174
11.3 Geometric Sequences and Series 176

11.1 Introduction to Sequences and Series

Sequences

A sequence is a collection of mathematical objects (in this class, numbers) in a specific order. Unlike in sets, the numbers in a sequence may be repeated.

Example 1 The sequence of all positive odd integers less than 20, in descending order, is

$$19, 17, 15, 13, 11, 9, 7, 5, 3, 1$$

The individual entries in a sequence are known as terms. Each term can be identified using a lower case letter (we'll typically use a) with a subscript indicating its position in the sequence.

Example 2 Find each of the following for the sequence above.

$a_1 = 19$ $a_3 = 15$ $a_6 = 9$ $a_{10} = 1$

If a sequence ends after a certain number of terms, it is finite. Otherwise, it is infinite.

While any numbers can be placed in an order to form a sequence, we're particularly interested in sequences which can be formed using a rule.

Explicit Rules

An explicit rule calculates the value of each term using its position in the sequence.

Example 3 Calculate the first 6 terms of the sequence $a_n = n^2 + 1$.

$$2, 5, 10, 17, 26, 37, \ldots$$

n	calculation	a_n
1	$(1)^2 + 1$	2
2	$(2)^2 + 1$	5
3	$(3)^2 + 1$	10
4	$(4)^2 + 1$	17
5	$(5)^2 + 1$	26
6	$(6)^2 + 1$	37

Recursive Rules

The word recursion refers to definitions or processes which refer to themselves in some way. A recursive rule calculates the value of each term using the values of the previous term, or possibly multiple previous terms.

If we think of a_n as the current term, then a_{n-1} is the previous term, and a_{n+1} is the next term.

These rules require at least one base case, a term that isn't defined recursively.

Example 4 Calculate the first 6 terms of the sequence $a_n = 2a_{n-1} - 3$, with $a_1 = 5$.

$$5, 7, 11, 19, 35, 67, \ldots$$

n	calculation	a_n
1		5
2	$2(5) - 3$	7
3	$2(7) - 3$	11
4	$2(11) - 3$	19
5	$2(19) - 3$	35
6	$2(35) - 3$	67

Example 5 List the first 10 terms of the Fibonacci sequence, defined as $f_n = f_{n-2} + f_{n-1}$, with $f_1 = f_2 = 1$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Types of Sequences

An arithmetic sequence has a constant difference between consecutive terms:

$$d = a_{n+1} - a_n$$

A geometric sequence has a constant ratio between consecutive terms:

$$r = \frac{a_{n+1}}{a_n}$$

Example 6 Determine whether the following sequences are arithmetic, geometric or neither.

$1, 5, 9, 13, 17, 21, \ldots$ arithmetic, as $d = 4$

$12, 6, 3, 1.5, 0.75, 0.375, \ldots$ geometric, as $r = \frac{1}{2}$

$1, 2, 6, 24, 120, 720, \ldots$ neither, as $6 - 2 \neq 2 - 1, \frac{6}{2} \neq \frac{2}{1}$

$8, 8, 8, 8, 8, 8, \ldots$ both, as $d = 0, r = 1$

Sums and Sigma Notation

Recall that the sum of a collection of numbers is the result obtained by adding them.

Example 7 Find the sum of 2, 4, 6, 8, 10 and 12.

$$2 + 4 + 6 + 8 + 10 + 12 = 42$$

We can write this sum more concisely using the upper case Greek letter sigma, Σ.

$$\sum_{k=1}^{6} 2k = 42$$

- Below Σ, we have the indexing variable, k, and its start value, 1.
- Above Σ, we have the end value of the indexing variable, 6.
- After Σ, we have the quantity to be summed, which is double the indexing variable in this case.

Example 8 Evaluate $\sum_{k=1}^{5} k^2$.

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55\end{aligned}$$

Example 9 Write $5 + 10 + 15 + 20 + \cdots + 100$ using sigma notation.

$$\sum_{k=1}^{20} 5k$$

Series

A series is the sum of the first n terms of a sequence[1], which can be written as

$$S_n = \sum_{k=1}^{n} a_k$$
$$= a_1 + a_2 + \cdots + a_n$$

Example 10 For $a_n = 3n + 5$, find S_8.

$S_8 = 148$

n	calculation	a_n	S_n
1	$3(1)+5$	8	8
2	$3(2)+5$	11	19
3	$3(3)+5$	14	33
4	$3(4)+5$	17	50
5	$3(5)+5$	20	70
6	$3(6)+5$	23	93
7	$3(7)+5$	26	119
8	$3(8)+5$	29	148

Example 11 For $a_n = 4a_{n-1} - 7$ with $a_1 = 3$, find S_5.

$S_8 = 239$

n	calculation	a_n	S_n
1		3	3
2	$4(3)-7$	5	8
3	$4(5)-7$	13	21
4	$4(13)-7$	45	66
5	$4(45)-7$	173	239

[1]Mathematicians usually call this a *partial sum*, and reserve the word *series* for an infinite sum.

11.2 Arithmetic Sequences and Series

Recall that an arithmetic sequence has a constant difference between consecutive terms:

$$d = a_{n+1} - a_n$$

Theorem

The recursive rule for an arithmetic sequence with difference d is

$$a_n = a_{n-1} + d$$

Example 1 Find the recursive rule for the sequence $5, 2, -1, -4, -7, \ldots$

$$a_n = a_{n-1} - 3, \quad a_1 = 5$$

Example 2 An arithmetic sequence begins with -2 and 4. State its recursive rule and find the first 8 terms of the sequence.

$$d = 6 \implies a_n = a_{n-1} + 6, \quad a_1 = -2$$

$$-2, 4, 10, 16, 22, 28, 34, 40, \ldots$$

We can use the recursive rule repeatedly to find expressions for the terms following a_1.

$$a_2 = a_1 + d \qquad \begin{aligned} a_3 &= a_2 + d \\ &= a_1 + 2d \end{aligned} \qquad \begin{aligned} a_4 &= a_3 + d \\ &= a_1 + 3d \end{aligned} \qquad \begin{aligned} a_5 &= a_4 + d \\ &= a_1 + 4d \end{aligned}$$

Theorem

The explicit rule for an arithmetic sequence with difference d and first term a_1 is

$$a_n = (n-1) \cdot d + a_1$$

The related function $f(n) = a_n$ is linear.

Example 3 Find the 50th term of the sequence $1, 5, 9, 13, 17, \ldots$

$$a_1 = 1, \quad d = 4 \implies a_n = (n-1)\cdot 4 + 1$$
$$\implies a_{50} = 49 \cdot 4 + 1 = 197$$

Example 4 In the sequence $a_n = a_{n-1} - 9$, $a_1 = 500$, which term is equal to 221?

$$d = -9 \implies a_n = (n-1)\cdot(-9) + 500 = 221$$
$$-9(n-1) = -279$$
$$n - 1 = 31$$
$$n = 32$$

So the 32nd term of the sequence is 221.

Theorem

The finite series of an arithmetic sequence given by a_n is

$$S_n = n \cdot \frac{a_1 + a_n}{2}$$

Example 5 For $a_n = a_{n-1} - 4$, $a_1 = 88$, find the sum of the first 40 terms.

$$d = -4$$

$$a_n = (n-1)(-4) + 87$$
$$a_40 = 39(-4) + 88$$
$$= -68$$

$$S_{40} = 40 \cdot \frac{a_1 + a_40}{2}$$
$$= 40 \cdot \frac{88 - 68}{2}$$
$$= 40 \cdot 10$$
$$= 400$$

Example 6 Find the sum of the odd numbers between 0 and 200.

$$a_1 = 1, \; d = 2$$
$$a_n = (n-1)\cdot 2 + 1 = 199$$
$$2(n-1) = 198$$
$$n - 1 = 99$$
$$n = 100$$

$$S_{100} = 100 \cdot \frac{1 + 199}{2}$$
$$= 100 \cdot 100$$
$$= 10000$$

11.3 Geometric Sequences and Series

Recall that a geometric sequence has a constant ratio between consecutive terms:

$$r = \frac{a_{n+1}}{a_n}$$

Theorem

The recursive rule for a geometric sequence with ratio r is

$$a_n = r \cdot a_{n-1}$$

Example 1 Find the recursive rule for the sequence $\frac{1}{18}, \frac{1}{3}, 2, 12, 72, \ldots$

$$a_n = 6a_{n-1}, \quad a_1 = \frac{1}{18}$$

Example 2 An geometric sequence begins with -2 and 4. State its recursive rule and find the first 8 terms of the sequence.

$$r = -2 \implies a_n = -2a_{n-1}, \quad a_1 = -2$$

$$-2, 4, -8, 16, -32, 64, -128, 256, \ldots$$

We can use the recursive rule repeatedly to find expressions for the terms following a_1.

$$a_2 = r \cdot a_1 \qquad \begin{aligned} a_3 &= r \cdot a_2 \\ &= r^2 \cdot a_1 \end{aligned} \qquad \begin{aligned} a_4 &= r \cdot a_3 \\ &= r^3 \cdot a_1 \end{aligned} \qquad \begin{aligned} a_5 &= r \cdot a_4 \\ &= r^4 \cdot a_1 \end{aligned}$$

Theorem

The explicit rule for a geometric sequence with ratio r and first term a_1 is

$$a_n = a_1 \cdot r^{n-1}$$

The related function $f(n) = a_n$ is exponential.

Example 3 Find the 12th term of the sequence $640, 320, 160, 80, \ldots$

$$a_1 = 640, \quad r = \tfrac{1}{2} \implies a_n = 640 \cdot \left(\tfrac{1}{2}\right)^{n-1}$$
$$\implies a_{12} = 640 \cdot \left(\tfrac{1}{2}\right)^{11} = \tfrac{5}{16}$$

Example 4 Which term of the sequence $a_n = 5a_{n-1}$, $a_1 = 3$ is the first to be greater than 1 billion?

$$\begin{aligned} r = 5 \implies a_n = 3 \cdot 5^{n-1} &> 10^9 \\ 5^{n-1} &> \tfrac{10^9}{3} \\ n - 1 &> \log_5\left(\tfrac{10^9}{3}\right) = 12.19 \\ n &> 13.19 \\ a_{14} &= 3.662 \times 10^9 \end{aligned}$$

Theorem

The finite series of a geometric sequence given by a_n is

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$$

Example 5 For $a_n = \frac{1}{2}a_{n-1}$, $a_1 = 100$, find the sum of the first 8 terms.

$$\begin{aligned} r &= \tfrac{1}{2} \\ S_{10} &= 100 \cdot \frac{1 - \left(\frac{1}{2}\right)^8}{1 - \frac{1}{2}} \\ &= 199.22 \end{aligned}$$

Example 6 If the sum of the first 4 terms of $a_n = 3a_{n-1}$ is 480, what are those 4 terms?

$$\begin{aligned} r &= 3 \\ S_4 &= a_1 \cdot \frac{1 - 3^4}{1 - 3} \\ &= 40a_1 = 480 \end{aligned}$$
$$a_1 = 12, \quad a_2 = 36, \quad a_3 = 108, \quad a_4 = 324$$

Chapter 12

Data and Statistics

12.1 Statistical Concepts 180
12.2 Normal Distributions 183
12.3 Bivariate Data 185
12.4 Collecting and Presenting Data 188

12.1 Statistical Concepts

In the field of statistics, a variable is a characteristic of a person or thing, which can have different values for each person or thing. A recorded value of a variable is called a datum, the plural of which is data. The two main types of variables are

- quantitative variables, whose data are numerical values for which it makes sense to use with arithmetic operations, and
- categorical variables, whose data place the people or things into groups or categories.

In this class, we'll mostly focus on quantitative variables and data.

Example 1 Decide if the following are quantitative or categorical.

- The salary of a software engineer. quantitative
- The fur color of a pet cat. categorical
- The zip code of a customer. categorical
- The weight of a football player. quantitative
- The number of students in an Algebra 2 class. quantitative

In this section, we'll focus on univariate data, which is data for a single variable.

A statistic is a single measure which summarizes a characteristic of a collection of data.

Measures of Central Tendency

A measure of central tendency is a statistic which uses a single number to represent an entire set of data.

- The mean is the sum of the data values divided by their number: $\bar{x} = \frac{\text{total}}{\text{count}} = \frac{\sum x}{n}$
- The median is the value in the middle when the data are ordered, or the mean of the middle two values.
- The mode is the most frequent value.

Example 2 Find the mean, median and mode of 2, 3, 3, 3, 4, 7, 7 and 11.

$$\text{mode} = 3$$
$$2, 3, 3, 3, 5, 7, 7, 11$$
$$\text{median} = 3.5$$

$$\begin{aligned}\bar{x} &= \frac{2+3+3+3+4+7+7+11}{8}\\ &= \frac{40}{8}\\ &= 5\end{aligned}$$

Measures of Spread

A measure of spread is a statistic which indicates how far the data deviates from the center.

- The variance measures spread using the differences of each value from the mean, and is calculated with the formula:

$$s^2 = \frac{\sum(x-\bar{x})^2}{n-1}$$

- The standard deviation is the square root of the variance, and is used more often as it shares the same units as the data:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

- The range is the difference between the smallest and largest values.
- The interquartile range, or IQR, is the difference between Q_1 and Q_3, which are the medians of the lower and upper halves of the data respectively.

Example 3 Find the standard deviation of the values in the previous example.

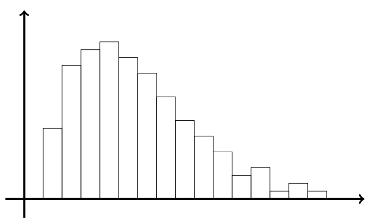
$$\bar{x} = 5$$
$$\sum(x-\bar{x})^2 = 66$$
$$\begin{aligned}s &= \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}\\ &= \sqrt{\frac{66}{7}}\\ &= 3.071\end{aligned}$$

x	$x-\bar{x}$	$(x-\bar{x})^2$
2	-3	9
3	-2	4
3	-2	4
3	-2	4
4	-1	1
7	2	4
7	2	4
11	6	36

Skewed Distributions

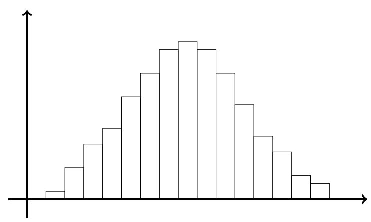
Examining a histogram representing a set of univariate data can reveal characteristics of the data.

If the bulk of the data is situated toward one end of its range, the data is said to be skewed. The direction of the skewness is the same as the direction of the distribution's tail.



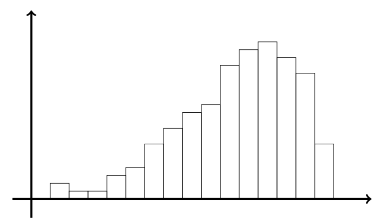
positively skewed
or right-skewed

mean > median



symmetric
not skewed

mean ≈ median



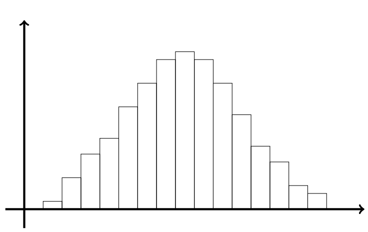
negatively skewed
or left-skewed

mean < median

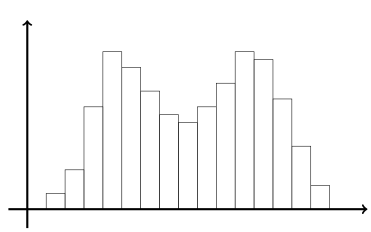
The mean is affected by skewed values more than other measures of central tendency, so the relationship between mean and median can indicate the direction of any skewness.

Unimodal and Multimodal Distributions

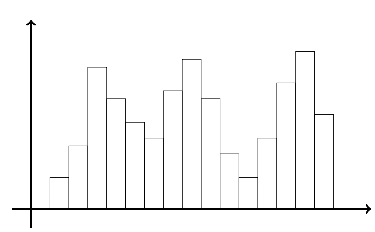
Data distributions can also be characterized by the number of peaks. It is typical to use the suffix modal to refer to these, even if the peaks do not have the same height, and therefore do not strictly meet the definition of the mode.



unimodal



bimodal



trimodal

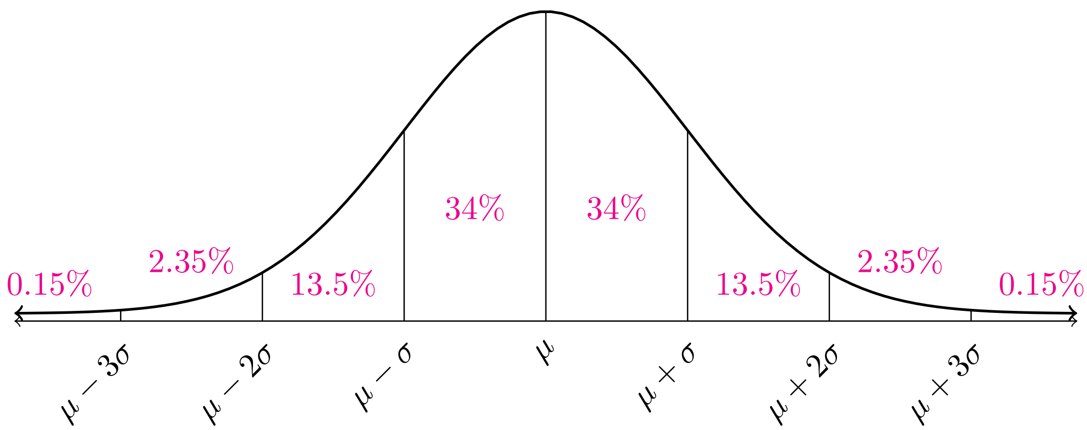
Distributions with more than one peak can also be called multimodal.

12.2 Normal Distributions

A normal distribution is a type of probability distribution. Each normal distribution is defined by two parameters:

- The mean, represented by μ (lower case Greek letter mu).
- The standard deviation, represented by σ (lower case Greek letter sigma).

The normal distribution can be graphed using a normal curve, which is sometimes called a bell-shaped curve. The area under the curve can be interpreted as probabilities in the related normal distribution.



- The distribution is unimodal, as it has one mode at the mean.
- The distribution is symmetric about the mean. 50% of the area is less than the mean, and 50% is greater than the mean.
- The **68-95-99.7 rule** states that
  - about 68% of the area is within one standard deviation of the mean,
  - about 95% of the area is within two standard deviations of the mean, and
  - about 99.7% of the area is within three standard deviations of the mean.

If a univariate data set is unimodal and symmetric, then it may be appropriate to use a normal distribution to model the data. We can fit the distribution to the data by choosing parameters

$$\mu = \bar{x} \qquad \sigma = s$$

Note the different symbols for mean and standard deviation. While we often choose them to have the same values, they have different meanings. $\bar{x}$ and s are the statistics calculated from the data, while μ and σ are the parameters of the distribution.

If X is a random variable, then we can use the notation

$$P(a < X < b)$$

to represent:

- The proportion of individuals whose values which fall between a and b.
- The probability that an individual chosen at random has a value between a and b.

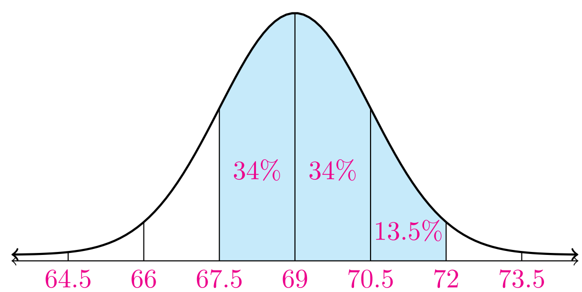
Example 1 The heights of a group of students are normally distributed with a mean of 5 ft 9 in and a standard deviation of 1.5 in.

a) Find the proportion of students whose heights are between 5 ft 7.5 in and 6 ft.

Let X be the height of a student.

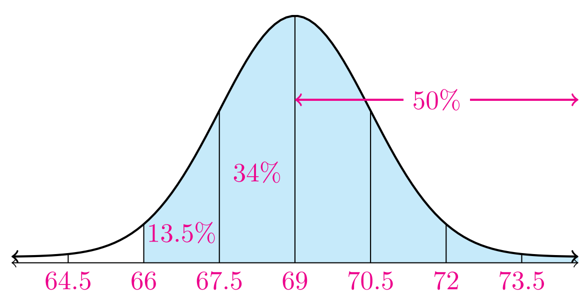
$\mu = 69\,\text{in} \quad \sigma = 1.5\,\text{in}$

$$\begin{aligned} P(67.5 < X < 72) &= 68\% + 13.5\% \\ &= 81.5\% \end{aligned}$$



b) Find the probability that a randomly chosen student is taller than 5 ft 6 in.

$$\begin{aligned} P(X > 66) &= 13.5\% + 34\% + 50\% \\ &= 97.5\% \end{aligned}$$

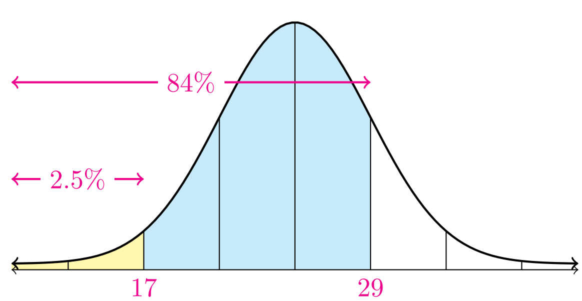


Example 2 In a normally distributed data set, 84% of the data values are less than 29, and 2.5% of the data values are less than 17. What are the mean and standard deviation?

$$\begin{cases} \mu + \sigma = 29 \\ \mu - 2\sigma = 17 \end{cases}$$

Subtracting the equations gives $3\sigma = 12$

$\implies \sigma = 4 \implies \mu = 25$



12.3 Bivariate Data

When data is collected for two variables from the same set of subjects, it is called bivariate data. In these cases, our interest is in knowing if there is an association between the variables, which means that changes in one variable tend to occur with changes in the other.

Review of Regression

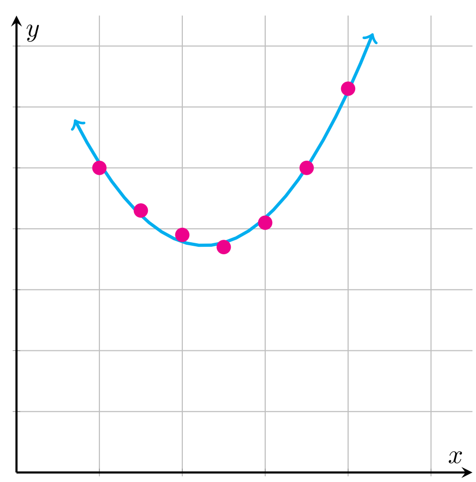
A key tool we have for examining bivariate data is regression, as we've studied previously. While we've used linear, quadratic and exponential regression, and we'll continue to restrict ourselves to those three for this class, regression is possible using any type of function for which an association could exist.

Recall:

- The aim of regression is to find a function which models an association between variables.
- The coefficient of determination, denoted by R^2, is a number between 0 and 1 indicating how well the model fits the data, with $R^2 = 1$ indicating a perfect fit.
- The correlation coefficient, denoted by r, is a number between -1 and 1 which indicates the strength and direction of the linear association between the two variables. For linear regression, $R^2 = r^2$.

Example 1 Find a function to model the data below.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	5.0	4.3	3.9	3.7	4.1	5.0	6.3



Shape formed by points suggests quadratic.

Using quadratic regression,

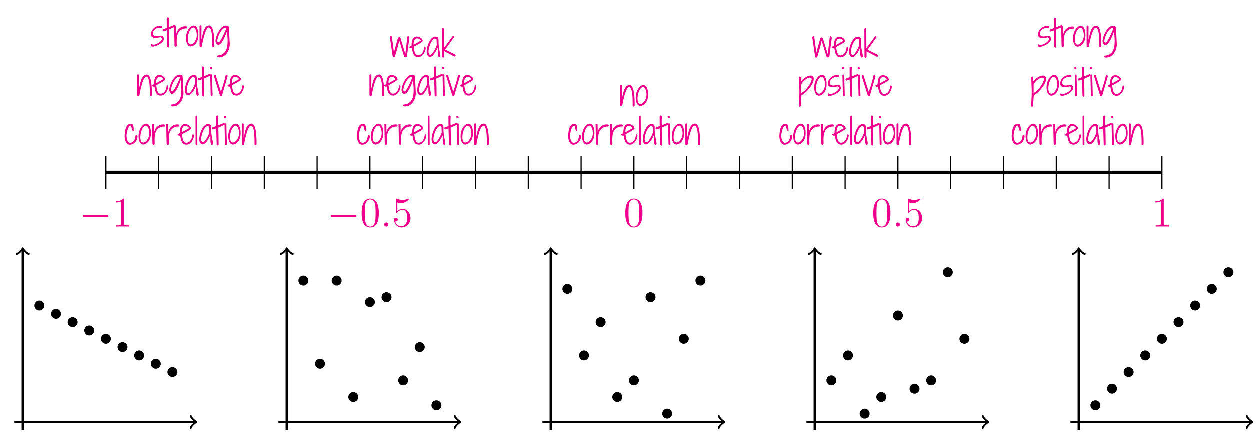
$R^2 = 0.992$ indicates good fit.

$f(x) = 0.84x^2 - 3.82x + 8.06$

Correlation and Causation

Correlation measures a linear relationship between variables by indicating how one variable changes as the other variable increases.

If increases in one variable sees proportionally similar increases in the other, there is a strong positive correlation between the variables, and r is close to 1. If increases in one variable sees proportionally similar decreases in the other, there is a strong negative correlation between the variables, and r is close to -1. In both cases, there is a strong linear association between the variables.



Suppose that there are two variables, X and Y, which have a strong positive correlation. As stated above, this means that as X increases, Y also increases at a proportionally similar rate. This does not mean, however, that an increase in X causes an increase in Y. There are actually three possibilities:

- Changes in X do indeed cause changes in Y.
- The causation is reversed, and changes in Y cause changes in X.
- Changes in X and Y are both caused by changes in a third variable.

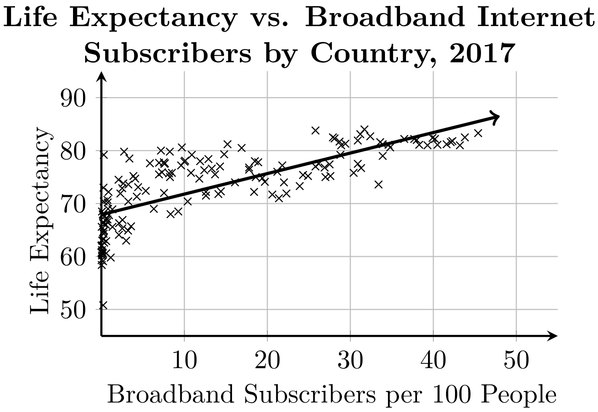
Not understanding this (or deliberately ignoring this) leads many people to make false claims not supported by the data. As you hear or read statistical conclusions made by others, or are trying to draw your own conclusions, it is vital to remember this principle:

Correlation vs. Causation

Correlation does not imply causation.

Example 2 This graph and the correlation coefficient $r = 0.7485$ show that there is a fairly strong positive correlation between the number of broadband internet subscriptions in a country and the life expectancy in that country.

Is it reasonable to say that if a country wants to raise life expectancy, they should improve their internet infrastructure?



Sources:
https://data.worldbank.org/indicator/IT.NET.BBND.P2
http://gapm.io/ilex

No, as the correlation does not imply that broadband internet causes an improved life expectancy. It is more likely that increases in both variables are caused by increases in the wealth of the country.

Discrete and Continuous Models

A quantitative variable which can take only distinct, countably-many values is called discrete. These values generally arise from a counting process.

A quantitative variable which can take any value within an interval is called continuous. These values generally arise from a measuring process.

Distinguishing between the two is important for deciding how to create graphs modeling the variable.

Example 3 A local car dealer promises to sponsor the high school softball team \$500, plus \$150 for each run they score in the next game, up to a total sponsorship of \$2000. Create a graph relating sponsorship money to runs scored.

Independent Variable: runs scored

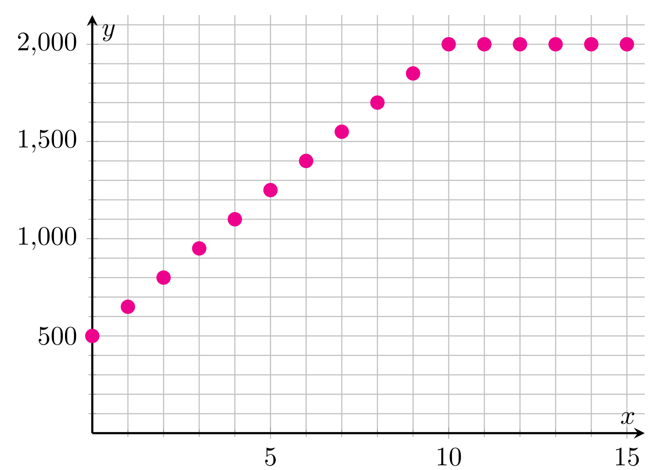
Dependent Variable: sponsorship money

Discrete/Continuous: discrete

Domain: $\{0, 1, 2, \ldots\}$

Function:

$$f(x) = \begin{cases} 150x + 500 & x = 0, 1, \ldots, 10 \\ 2000 & x = 11, 12, \ldots \end{cases}$$



12.4 Collecting and Presenting Data

The aim of statistics is to understand truths about the world through the collection and interpretation of data. Every day, people form beliefs and make decisions based on the data that have been presented to them.

Unfortunately, data can be collected in ways that make them unreliable, or can be presented in ways that are misleading. While some people will manipulate data in these ways deliberately, it is very easy to accidentally misuse data. Knowing how data can be misinterpreted helps us to avoid being deceived by claims made by others, and to better understand the data we collect ourselves.

Populations and Samples

If we're interested in data regarding a particular class of people or things, the population is the entire set of people or things in that class.

Example 1 A medical researcher is collecting data about the weights of 15 year olds in Oklahoma. What is the population?

The population is the set containing every 15 year old in Oklahoma.

If data are collected from every individual in the population, the process is called a census. This is ideal, as we know that the data truly represents the entire population. However, doing so is often impractical.

Instead, data are typically collected from a sample, which is a subset of the population which is intended to represent the entire population. The sample should contain a large number of individuals to minimize the effect of random variation.

There are many different methods to select the sample, with varying quality. Here are a few common sampling methods:

- A simple random sample selects the members of the sample from the entire population at random. This is usually best practice if possible. This can be as simple as drawing names from a hat, or can be done by assigning numbers to each individual and using a random number generator.
- A stratified sample places individuals into groups, then randomly selects members from every group. This ensures that every group is represented in the sample.
- A clustered sample places individuals into groups, then selects every member from randomly selected groups. This is often easier to administer, while still containing some randomness in the sample.

- A voluntary response sample selects individuals who are willing to participate in a survey. Sometimes this is the only way to collect data, for legal or ethical reasons, but may introduce sample bias.

- A convenience sample selects the individuals who are easiest to collect data from. This almost certainly introduces sample bias. While this is a popular method because it is easy, informed statisticians should not use it.

Any factor that affects the data in a way such that they do not represent the true state of the population is called a bias. If the source of the bias is the way the sample was selected, it is called sample bias. Other biases include observer bias, which is where the presence of an observer affects the behavior or response of individuals in the sample.

Example 2 A business manager at a large company is concerned that many of her employees are spending a lot of time using social media when they should be working. She asks her assistant manager to conduct some research. He asks the first five people into the office the next day how much time they've wasted on social media. He reports to his boss that there is no social media problem at the company.

Are there any issues regarding the data collection in this scenario?

Small sample: In a large company, five people is not representative of the population of employees.

Convenience sample: The assistant manager didn't use random sampling at all. It may be the case that these employees are earliest because they are relatively busy, and have less time to waste.

Observer bias: Employees are unlikely to admit to management the amount of time they've wasted online when they should have been working.

Recognizing Distorted Data Displays

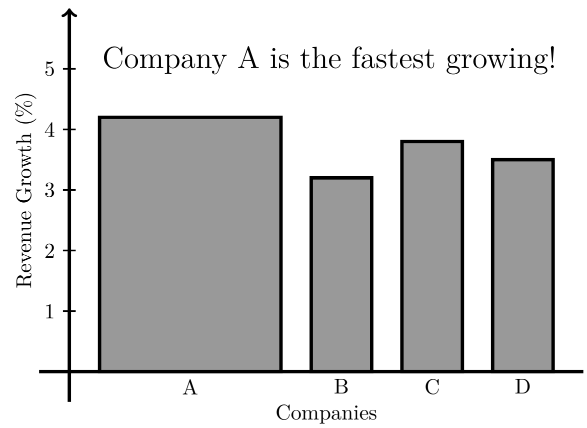
Presenting data in a graph is a useful way to communicate and emphasize aspects of the data that are important to the author of the display. Unfortunately, it is possible to present data in ways that, while not false, are misleading.

An important rule to remember when presenting data is the area principle. This says that if a quantity is represented by a two-dimensional region in a graph, the area of the region should be proportional to the quantity.

Example 3

This chart violates the area principle because the bars do not have the same width. Even though Company A does have the highest growth, the difference in growth appears to be much greater because the bar's area is much greater.

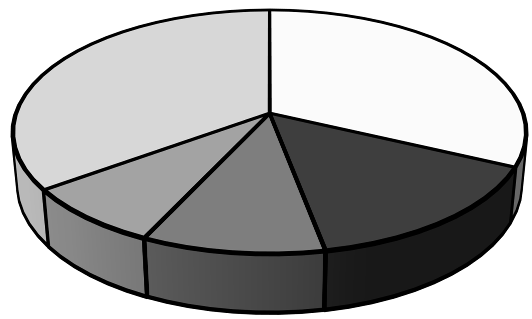
In general, the bars in a bar chart should all have the same width.



Example 4

This chart violates the area principle, because the 3D effect on the pie chart causes some of the sectors to have additional visible area along the edge.

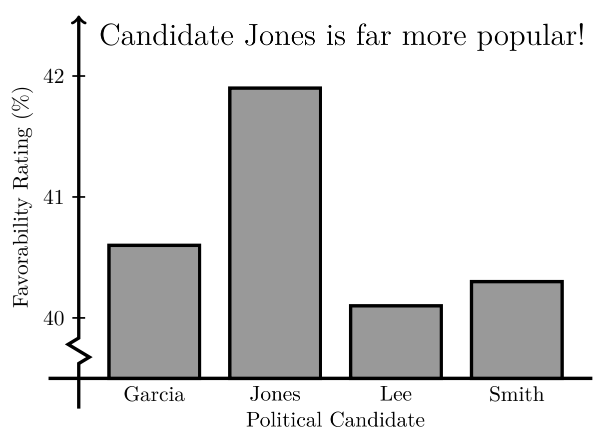
While they might look clever, using 3D effects in data displays should always be avoided.



Example 5

This chart violates the area principle because the lengths of the bars are not proportional to their corresponding value. Even though Jones does have the highest favorability, the difference in favorability appears to be much greater because the bar's area is much greater.

This occurs because the scale on the y-axis has been distorted.



A graph such as a line chart can also have a distorted y-axis. In some cases, this is justified when seeing trends and small changes is important, such as in financial charts. In general, however, readers will expect a linear scale beginning at zero.